Internship Report – MPRI M2
On First-order Combinatorial Proofs

Jui-Hsuan Wu, supervised by Lutz Straßburger, Inria Saclay

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General Context

The notion of combinatorial proofs has been introduced by D. Hughes to give a syntax-free representation of proofs in classical propositional logic. This gives rise to a new approach to the proof identity problem: "we say that two proofs are the same if they correspond to the same combinatorial proof". Combinatorial proofs have also been introduced in different logical systems such as relevant logic and modal logics.

Problem Studied

Recently, D. Hughes introduced a version of combinatorial proofs for first-order classical logic. However, there seem to be a more compact way to represent these combinatorial proofs. During the internship, I established a relation between fonets, an essential part of combinatorial proofs, and unification nets, a notion introduced for representing proof nets without certain redundancies. This relation gives us the idea of proving the completeness and correctness of first-order combinatorial proofs in a simpler way.

Proposed Contributions

The contributions of my internship are the following: establishing the relation between fonets and unification nets and giving a new proof of completeness of first-order combinatorial proofs.

Arguments Supporting Their Validity

This gives a shorter and more compact proof of completeness of first-order combinatorial proofs.

Summary and Future Work

After working with the completeness, it should be natural to apply the same idea to the correctness of first-order combinatorial proofs. We want to establish the equivalence between weakening/contraction derivations and skew bifibrations. However, the results from the propositional case might not be sufficient and this problem is much more subtle.

1 Introduction

The notion of combinatorial proofs has been introduced by Dominic Hughes to give a presentation for proofs in classical propositional logic independent from syntactic proof systems.

My internship aims at giving a simpler proof of the completeness and soundness of first-order combinatorial proofs by establishing a relation between first-order combinatorial proofs and the sequent calculus **LK**. Before the paper introducing first-order combinatorial proofs, D. Hughes introduced unification nets which give a compact representation of proof nets for MLL1 that eliminates unnecessary redundancies. I establish the equivalence between unification nets and first-order nets introduced in [4].

2 First-order combinatorial proofs and Deep inference

2.1 First-order logic

Definition 2.1. Terms and atoms are generated inductively:

- if f is an n-ary function (resp. predicate) symbol and $t_1, \cdots, t_n$ are terms, then $ft_1 \cdots t_n$ is a term (resp. atom).

We also extend the set of atoms with the logical constants 1 and 0.

Definition 2.2. Formulas are generated inductively:

- atoms are formulas,
- if ϕ and θ are formulas, then $\phi \vee \theta$ and $\phi \wedge \theta$ are also formulas,
- if ϕ is a formula and x a variable, then $\exists x\phi$ and $\forall x\phi$ are formulas.

We assume that each predicate symbol p has a dual predicate symbol $\bar{p}$ such that $\bar{p} \neq p$ and $\bar{\bar{p}} = p$. We can thus extend the duality to atoms (atomic formulas) with $\overline{pt_1 \cdots t_n} = \bar{p}t_1 \cdots t_n$, $\overline{0} = 1$ and $\overline{1} = 0$.

We now define $\neg$ and $\Rightarrow$ as follows.

Definition 2.3.

- $\neg(\alpha) = \overline{\alpha}$ for each atom α
- $\neg(\phi \vee \theta) = (\neg\phi) \wedge (\neg\theta)$, $\neg(\phi \wedge \theta) = (\neg\phi) \vee (\neg\theta)$
- $\neg\forall x\phi = \exists x\neg\phi$, $\neg\exists x\phi = \forall x\neg\phi$

We define $\Rightarrow$ by $\phi \Rightarrow \theta = (\neg\phi) \vee \theta$.

Definition 2.4. We say a formula is rectified if all bound variables are distinct from one another and from all free variables. For example, $(px \wedge \exists y\ qy) \vee \forall z\ rz$ is rectified but $(px \wedge \exists x\ qx) \vee \forall y\ ry$ is not.

2.2 Graphs

A directed graph (V, E) is a finite set V of *vertices* and a set $E \subseteq V \times V$ of *edges* on V. An undirected graph (V, E) is a finite set V of *vertices* and a set E of directed edges, i.e., two-element subsets of V. For a given (directed or undirected) graph G, we write V_G and E_G for its vertex and edge sets. We write vw for an edge (v, w) or $\{v, w\}$.

Let $G = (V, E)$ and $G' = (V', E')$ be two undirected graphs. Without loss of generality, we assume that the two sets of vertices are disjoint. We now define some operations on them.

Definition 2.5. The *union* $G + G'$ is defined as the graph $(V \cup V', E \cup E')$

Definition 2.6. The *join* $G \times G'$ is defined as the graph $(V \cup V', E \cup E' \cup \{vv' | v \in V, v' \in V'\})$

Definition 2.7. A cograph $G = (V, E)$ is a P_4-free undirected graph, i.e., for any distinct vertices $v_1, v_2, v_3, v_4 \in V$, the restriction of edges on them is not equal to $\{v_1v_2, v_2v_3, v_3v_4\}$

Definition 2.8. Given a graph $G = (V, E)$, a set $W \subseteq V$ induces a matching in G if for all $w \in W$, there exists a unique $w' \in W$ such that $ww' \in E$.

Definition 2.9. In a graph $G = (V, E)$, the *neighbourhood* $N(v)$ of $v \in V$ is defined as the set $\{w|vw \in E\}$, and a *module* is a set $M \subseteq V$ such that $N(v)\backslash M = N(w)\backslash M$ for all $v, w \in M$. A module M is strong if for every module M', we have $M' \subseteq M$, $M \subseteq M'$ or $M' \cap M = \varnothing$.

2.3 First-order graphs

Definition 2.10. A cograph is *logical* if every vertex is labelled by an atom or variable, and it has at least one atom-labelled vertex. An atom-labelled vertex is called a *literal* and a variable-labelled vertex is called a *binder*. A binder labelled with x is called an x-binder. The scope of a binder b is the smallest strong module containing at least two vertices, including b.

Example 2.11. Here is an example of logical cograph $D =$

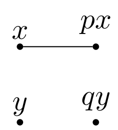

Figure 1. A logical cograph.

The scope of the x-binder contains only the vertex labelled by px and itself while the scope of the y-binder contains all the vertices.

Definition 2.12. In a graph $G = (V, E)$, a binder is *existential* (resp. *universal*) it is connected (resp. disconnected) to every other vertex in its scope.

In Example 2.11, the x-binder is existential and the y-binder is universal.

Definition 2.13. An x-binder is *legal* if its scope contains at least one literal and no other x-binder.

Definition 2.14. An *x-literal* is one whose atom contains the variable x. An x-binder *binds* every x-literal in its scope.

Definition 2.15. An x-binder is *rectified* if it is the only x-binder and its scope contains every x-literal. A fograph is *rectified* if all of its binders are rectified.

Definition 2.16. A *first-order graph* or *fograph* is a logical cograph whose binders are all legal.

It is clear that both binders in Example 2.11 are legal. Hence, D is a fograph.

Definition 2.17. The graph $G(A)$ of a formula A is defined inductively by:

$$G(a) = \cdot a \text{ for every atom } a$$

$$G(A \vee B) = G(A) + G(B) \qquad G(A \wedge B) = G(A) \times G(B)$$

$$G(\forall x A) = \cdot x + G(A) \qquad G(\exists x A) = \cdot x \times G(A)$$

Example 2.18. The graphs of the formulas $(\exists x\ px) \vee (\forall y\ qy)$ and $\forall y\ ((\exists x\ px) \vee qy)$ are equal to the fograph D in Example 2.11.

Remark 2.19. $G(A)$ is a fograph for all formula A.

Definition 2.20. The *binding graph* $\overrightarrow{G}$ of a fograph G is the directed graph $(V_G, \{(b, l) \mid b \text{ binds } l\})$.

Example 2.21. The binding graph $\overrightarrow{D}$ of the fograph D in Example 2.11 is the following:

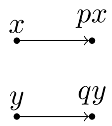

Figure 2. The binding graph of the fograph D.

2.4 Fonets

Definition 2.22. Two atoms are *pre-dual* if their predicate symbols are dual (e.g. pxy and $\bar{p}yz$) and two literals are *pre-dual* if their labels (atoms) are pre-dual.

Definition 2.23. A *linked fograph* is a coloured fograph such that:

- every colour, called a *link*, consists of two pre-dual literals, and
- every literal is $\mathtt{t}$-labelled or in a link.

Definition 2.24. Let G be a linked fograph. The set of links can be seen as a unification problem. A *dualizer* of G is an assignment unifying all the links of G.

Remark 2.25. We know that first-order unification is decidable, and there exists a most general unifier if the unification problem is solvable. Hence, we can define the notion of ”most general dualizer” of a linked fograph given.

Definition 2.26. Let G be a linked fograph. A *dependency* is a pair $(\cdot x, \cdot y)$ of an existential binder $\cdot x$ and a universal binder $\cdot y$ such that the most general dualizer assigns to x a term containing y. A *leap* is either a link or a dependency. The *leap graph* of G is the graph (V_G, L_G) where L_G is the set of leaps of G.

Definition 2.27. We say that a set W induces a *bimatching* in a linked fograph G if W induces a matching in G and induces a matching in the leap graph of G.

Now we define the notion of fonets.

Definition 2.28. A *fonet* or *first-order net* is a linked fograph which has a dualizer but no induced bimatching.

Example 2.29. Here is a fonet:

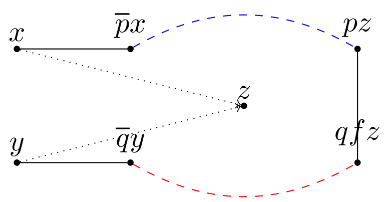

Figure 3. A fonet.

2.5 Skew bifibrations

Definition 2.30. A graph homomorphism $f : (V, E) \to (V', E')$ is a *fibration* if for all $v \in V$ and $wf(v) \in E'$, there exists a unique $\tilde{w}$ such that $\tilde{w}v \in E$ and $f(\tilde{w}) = w$.

Definition 2.31. An undirected graph homomorphism $f : (V, E) \to (V', E')$ is a *skew fibration* if for all $v \in V$ and $wf(v) \in E'$, there exists $\tilde{w}$ such that $\tilde{w}v \in E$ and $f(\tilde{w})w \notin E'$.

Definition 2.32. A *fograph homomorphism* is a graph homomorphism between the underlying graphs of the fographs considered.

Definition 2.33. A fograph homomorphism $f : G \to H$ is *label-preserving* if for all $v \in V_G$, the label of v equals the label of $f(v)$ in H, and is *existential-preserving* if for all existential binder b in G, the vertex $f(b)$ is an existential binder in H.

Definition 2.34. A *skew bifibration* is a label-preserving and existential-preserving fograph homomorphism $f : G \to H$ such that

- $f : G \to H$ is a skew fibration
- $f : \vec{G} \to \vec{H}$ is a fibration (on corresponding binding graphs).

Example 2.35. Here is a figure illustrating a skew bifibration:

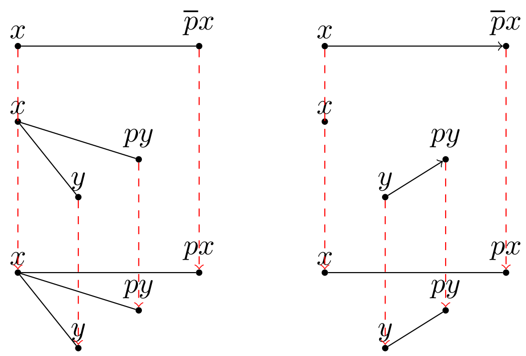

Figure 4. A skew bifibration (left) and its binding fibration (right). We use dashed lines for denoting the map.

2.6 Combinatorial proofs

Definition 2.36. A *combinatorial* proof of a fograph G is a skew bifibration $f : N \to G$ where N is a fonet.

Definition 2.37. A *combinatorial proof* of a formula ϕ is a combinatorial proof of its graph $G(\phi)$.

2.7 MLL1 and Unification nets

2.7.1 MLL1

In **MLL1**, terms and atoms are defined as the first-order logic.

Definition 2.38. Formulas are generated inductively:

- atoms are formulas,
- if ϕ and θ are formulas, then $\phi \parr \theta$ and $\phi \otimes \theta$ are formulas,

- if ϕ is a formula and x a variable, then $\exists x \phi$ and $\forall x \phi$ are formulas.

A formula ϕ is identified with its *formula tree* $F(\phi)$, a directed tree with leaves labelled by atoms and internal nodes labelled by connectives and quantifiers. A *sequent* is simply a disjoint union of formulas. We write comma for disjoint union.

Sequents are proved using the inference rules of **MLL1**:

$$\frac{}{\vdash A, \neg A}\ ax \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta}\ cut$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x A}\ \forall\ (x \notin fv(\Gamma)) \qquad \frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, \exists x A}\ \exists$$

We also consider the *mix* rule:

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$$

2.7.2 Unification nets

Definition 2.39. Given a sequent Γ in **MLL1** + *mix*, we define links as previously. A linking on Γ is a set of disjoint links whose union contains each atom and unit of Γ.

Definition 2.40. Let λ be a unifiable linking on a sequent $\Gamma = \phi_1, \cdots, \phi_n$ which can be seen as the formula $\phi = \phi_1 \parr \cdots \parr \phi_n$. The *unification structure* $U(\lambda)$ associated to λ is the the formula tree $F(\phi)$ together with an undirected edge between leaves l and l' for every link $\{l, l'\}$ in λ and a directed edge from $\exists x$ to $\forall y$ for every dependency $\exists x \to \forall y$.

Definition 2.41. A *switching graph* of a unification structure $U(\lambda)$ is any derivative of $U(\lambda)$ obtained by deleting all but one edge into each $\parr$ and $\forall$ and undirecting remaining edges.

Definition 2.42. A linking is correct if it is unifiable and all of the switching graphs of its associated unification structure are acyclic.

Definition 2.43. A (cut-free) unification net on a sequent Γ is a correct linking on Γ.

Example 2.44. Here is a unification net:

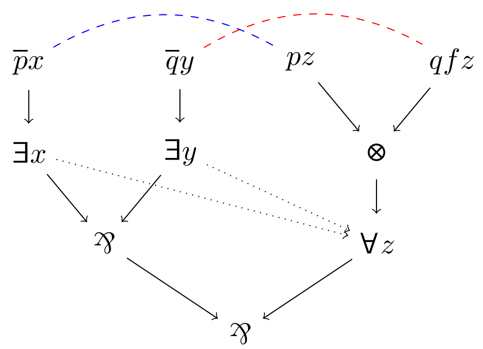

Figure 5. A unification net. (dashed lines = links and dotted lines = dependencies)

We also give some of its switching graph:

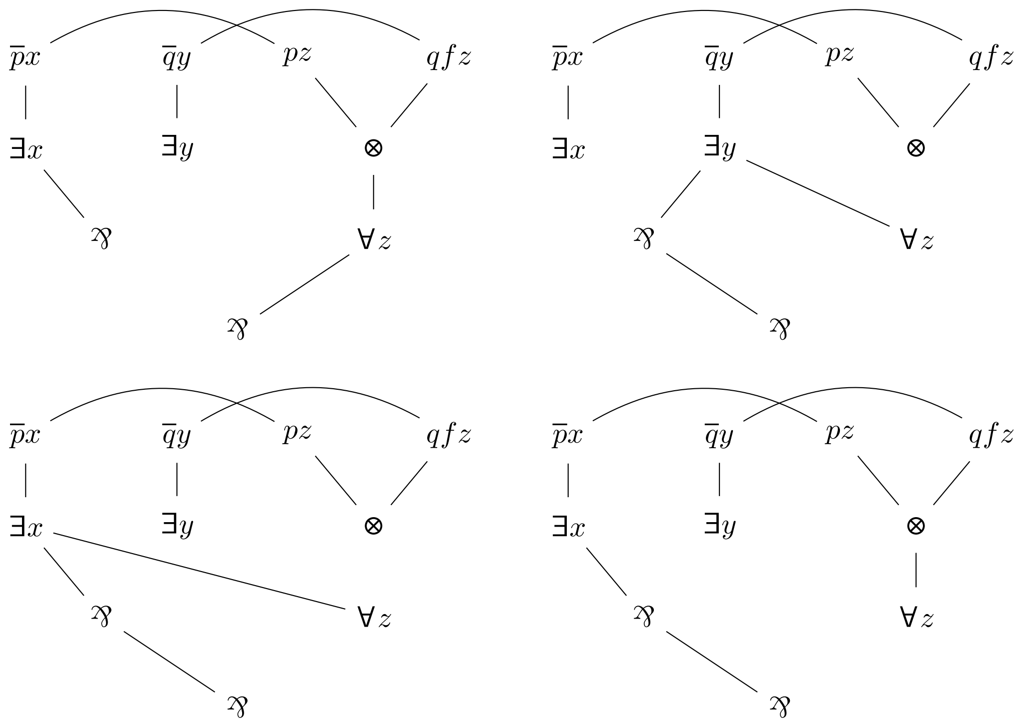

Figure 6. Some switching graphs of the unification net shown in Figure 5.

It is clear that they are all connected and acyclic.

2.8 Deep inference

In our study we also use some *deep inference* rules that are in the following form:

$$\frac{\vdash S\{A\}}{\vdash S\{B\}}$$

where $S\{\ \}$ stands for a *context*, which corresponds to a sequent or a formula with a hole taking the place of an atom, and $S\{A\}$ represents the sequent or formula obtained by replacing the hole in $S\{\ \}$ with the formula A.

$$C ::= \square \mid A \vee C \mid C \vee A \mid \exists x C \mid \forall x C$$

.

$$S ::= C|A, S|S, A$$

where A is a formula. The above rule can be thus seen as the rewriting rule $A \to B$.

Notation 2.45. We use the notation $\begin{array}{c} A \\ \| \mathcal{P} \\ B \end{array}$ for denoting that there is a derivation from premise $\vdash S\{A\}$ to conclusion $\vdash S\{B\}$ in system P for any context S.

3 From first-order logic to combinatorial proofs

3.1 LK

We start with Gentzen's sequent calculus system **LK**.

$$\frac{}{\vdash A, \neg A}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta}\ cut$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}\ \wedge \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee$$

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ ctr \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ wk$$

$$\frac{}{\vdash \mathsf{t}}\ \mathsf{t} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \mathsf{f}}\ \mathsf{f}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x A}\ \forall\ (x \notin fv(\Gamma)) \qquad \frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, \exists x A}\ \exists$$

Consider the following deep rules:

$$\frac{\vdash S\{A \vee A\}}{\vdash S\{A\}}\ \mathsf{c}\!\downarrow \qquad \frac{\vdash S\{\mathsf{f}\}}{\vdash S\{A\}}\ \mathsf{w}\!\downarrow$$

The , operator in the sequent should be interpreted as the disjunction $\vee$.

Note that the *ctr* (resp. *wk*) rule in **LK** is derivable in $\{\mathsf{c}\!\downarrow, \vee\}$ (resp. $\{\mathsf{w}\!\downarrow, \mathsf{f}\}$) and that $\mathsf{c}\!\downarrow$ and $\mathsf{w}\!\downarrow$ rules permute downwards with the non-structural rules of **LK**.

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ ctr \quad \rightsquigarrow \quad \cfrac{\cfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee A}\ \vee}{\vdash \Gamma, A}\ \mathsf{c}\!\downarrow$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, A}\ wk \quad \rightsquigarrow \quad \cfrac{\cfrac{\vdash \Gamma}{\vdash \Gamma, \mathsf{f}}\ \mathsf{f}}{\vdash \Gamma, A}\ \mathsf{w}\!\downarrow$$

We also give an example to show how rule permutation works:

$$\cfrac{\cfrac{\Gamma, A \vee A}{\Gamma, A}\ \mathsf{c}\!\downarrow \qquad \Delta, B}{\Gamma, \Delta, A \wedge B}\ \wedge \quad \rightsquigarrow \quad \cfrac{\cfrac{\Gamma, A \vee A \qquad \Delta, B}{\Gamma, \Delta, (A \vee A) \wedge B}\ \wedge}{\Gamma, \Delta, A \wedge B}\ \mathsf{c}\!\downarrow$$

We want to establish the following theorem:

Theorem 3.1. *Let Γ be a sequent. Then there is a proof of Π in* **LK** *iff there is a proof of some sequents Δ in* **MLL1** *and a derivation from Δ to Γ consisting of the $\mathsf{c}\!\downarrow$ and $\mathsf{w}\!\downarrow$ rules only.*

Proof. ($\Rightarrow$) This direction comes from the above observation: it suffices to permute downwards all the instances of the $\mathsf{c}\!\downarrow$ and $\mathsf{w}\!\downarrow$ rules.

($\Leftarrow$) We regard the proof in **MLL1** as a proof in **LK**. Then we put the dervation consisting of only $\mathsf{c}\!\downarrow$ and $\mathsf{w}\!\downarrow$ under the proof in **LK**. Now we try to permute all the instances $\mathsf{c}\!\downarrow$ and $\mathsf{w}\!\downarrow$ upwards with the rules of LK. For the $\mathsf{c}\!\downarrow$ part, the only non-trivial case is the permutation with the $\vee$ rule where the formula generated is $A \vee A$.

$$\dfrac{\dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A \vee A}\,\vee}{\vdash \Gamma, A}\,\mathsf{c}\!\downarrow \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,ctr$$

In this case, the permutation of this instance of $\mathsf{c}\!\downarrow$ stops and we continue with the remaining instances.

For the $\mathsf{w}\!\downarrow$ part, the only non-trivial case is the permutation with the f rule (or the instance of wk where f is introduced):

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, f}\,\mathsf{f}}{\vdash \Gamma, A}\,\mathsf{w}\!\downarrow \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma}{\vdash \Gamma, A}\,wk$$

In this case, the permutation of this instance of $\mathsf{w}\!\downarrow$ stops and we continue with the remaining instances.

□

In [3], D. Hughes proves the soundness and completeness of unification nets with respect to **MLL1**. In the following, we establish the equivalence between unification nets and fonets.

3.2 Equivalence between unification nets and fonets

In the following, we usually confound a vertex with its label.

We also confound $⅋$ with $\vee$ and $\otimes$ with $\wedge$ as unification nets and first-order nets (fonets) are defined in different contexts.

Definition 3.1. A *switching path* of a unification structure $U(\lambda)$ is a path in a switching graph of $U(\lambda)$.

Definition 3.2. A *switching path* of a formula tree $F(\phi)$ is a path in $F(\phi)$ that does not go through both incoming edges of a $⅋$.

Proposition 3.3. In a formula tree, the root is connected to every vertex by a switching path.

Now we give the key proposition relating a fograph to its corresponding formula tree.

Proposition 3.4. Let u and v be two distinct vertices of a fograph $G(\phi)$, then we have the equivalence between:

- u and v are adjacent in $G(\phi)$
- u and v are connected by a switching path of $F(\phi)$, and if one of them is a universal quantifier, then the other is not a descendant of the former.

Proof. By induction on ϕ.

- If ϕ is an atom, trivial.
- If $\phi = \phi_1 \wedge \phi_2$, then we distinguish two cases:
  - u and v are both in ϕ_1 (resp. ϕ_2): trivial by the induction hypothesis.
  - one of them is in ϕ_1 and the other is in ϕ_2: they are adjacent in $G(\phi)$ by definition. By Proposition 3.3, the one in ϕ_1 (resp. ϕ_2) is connected to the vertex representing ϕ_1 (resp. ϕ_2) by a switching path. Together with the two edges incident to $\phi_1 \wedge \phi_2$, we obtain a switching path connecting u and v.

- If $\phi = \phi_1 \vee \phi_2$, then we distinguish two cases:
  - u and v are both in ϕ_1 (resp. ϕ_2): trivial by the induction hypothesis.
  - one of them is in ϕ_1 and the other is in ϕ_2: they are not adjacent in $G(\phi)$ by definition. It is clear that they are not connected by a switching path.
- If $\phi = \exists x\ \phi'$, then we distinguish two cases:
  - u and v are both in ϕ': trivial by the induction hypothesis.
  - one of them is $\exists x$ and the other is in ϕ': trivial by Proposition 3.3
- If $\phi = \forall x\ \phi'$, then we distinguish two cases:
  - u and v are both in ϕ': trivial by the induction hypothesis.
  - one of them is $\forall x$ and the other is in ϕ': they are not adjacent in $G(\phi)$ by definition and it is clear that the former is a descendant of $\forall x$.

□

Proposition 3.5. If there exists an induced bimatching of the linked fograph $G = G(\phi)$, then there exists a switching graph of the corresponding unification net which contains a cycle.

Proof. Suppose that there exists a set W inducing a bimatching in G. Then (W, E_G) and (W, L_G) are matchings.
Let E_W (resp. L_W) be the restriction of E_G (resp. L_G) to W.
If $E_W \cap L_W \neq \emptyset$, then there exist u and v such that $uv \in E_G$ and $uv \in L_G$. By Proposition 3.4, there exists a switching path of the formula tree of ϕ. Together with the leap uv, this path induces a cycle in a switching graph of the corresponding unification structure.
We can now suppose that E_W and L_W are disjoint. It is not difficult to see the existence of an alternating and elementary cycle in the bicoloured graph $(W, E_W \uplus L_W)$, i.e. a cycle of which the edges are alternately in E_W and L_W and containing no two equal vertices. By Proposition 3.4, this cycle induces a cycle in the unification structure. Now we want to construct a switching graph that contains this cycle.
Consider a universal quantifier $\forall x$. If $\forall x \notin W$, then we keep the incoming edge from its direct subformula and remove all the dependencies. Otherwise, since (W, L_G) is a matching, there exists a unique existential quantifier adjacent to $\forall x$ and we keep thus the corresponding edge in the unification structure.
Now consider a $\invamp$. We distinguish three cases:

- the cycle goes through none of the two branches (incoming edges) of the $\invamp$: we can choose an arbitrary switching for this $\invamp$
- the cycle goes through exactly one branch: we choose the corresponding switching
- the cycle goes through both branches: this means that there exist $v_L \in W$ (resp. v_R) in the left (resp. right) branch, u_L, $u_R \in W$, such that $u_L v_L, u_R v_R \in E_W$ and that the corresponding switching path from u_L to v_L (resp. from u_R to v_R) goes through the left (resp. right) edge of $\invamp$.

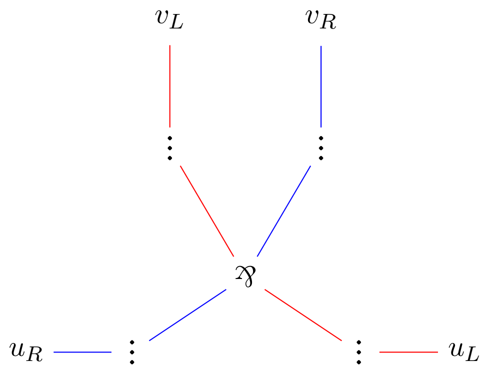

Figure 7. A schema showing that the two branches of the same ⅋ cannot be used in the cycle at the same time.

The red (resp. blue) path is the switching path corresponding to the edge $u_L v_L$ (resp. $u_R v_R$) in E_W.

It is clear that u_L (resp. u_R) is not in the branches of the ⅋. Otherwise, there will be no switching path from u_L to v_L

By Proposition 3.4, we know that u_L and u_R are not universal quantifiers which are ancestors the ⅋ and that there exist one switching path from u_L to v_L and one from u_R to v_R. In particular, there exist one switching path from u_L to the ⅋ and one from the ⅋ to v_R, and by concatenating the two, we obtain a switching path from u_L to v_R. By Proposition 3.4, u_L and v_R are thus adjacent in (W, E_G), which is impossible since (W, E_W) is a matching.

Notice that the switching paths here are in the underlying formula tree. We have to verify that they are compatible with the choices of switching made for universal quantifiers. That is, if $uv \in E_W$, then for all the universal quantifiers $\forall x$ on the switching path (in the formula tree), we have chosen in the first part of the proof to keep the only edge from the child of $\forall x$ to itself. In fact, if there exists a universal quantifier $w \in W$ on the switching path $u \to v$, then one of u and v is not a descendant of w. Moreover, if u (resp. v) is a universal quantifier, then w is not in its scope. By Proposition 3.4, $\{wu, wv\} \cap E_W \neq \varnothing$, which is impossible since (W, E_W) is a matching. We have thus constructed a switching graph containing this cycle. □

Proposition 3.6. If one of the switching graphs of the unification structure of ϕ contains a cycle or is not connected, then there exists an induced bimatching of the corresponding linked fograph.

Proof. We use frames introduced by Hughes in Section 4 of [3].

Definition 3.7. Let θ be a unification structure on an **MLL1** sequent Γ. We define the *frame* of θ by exhaustively applying the following subformula rewriting steps, to obtain a proof structure θ_m on an **MLL** sequent Γ_m:

1. **Encode dependencies as fresh links.** For each dependency $\exists x \to \forall y$, with corresponding subformulas $\exists x A$ and $\forall y B$, we add a fresh link as follows. Let P be a fresh (nullary) predicate symbol. Replace $\exists x A$ with $P \otimes \exists x A$ and $\forall y B$ with $\overline{P} ⅋ \forall y B$, and add an axiom link between P and $\overline{P}$.

2. **Erase quantifiers.** After step 1, erase all the quantifiers. (We no longer need their leaps since they are encoded as links in step 1.

3. **Simplify atoms.** After step 2, replace every predicate $Pt_1 \cdots t_n$ with a nullary predicate symbol P.

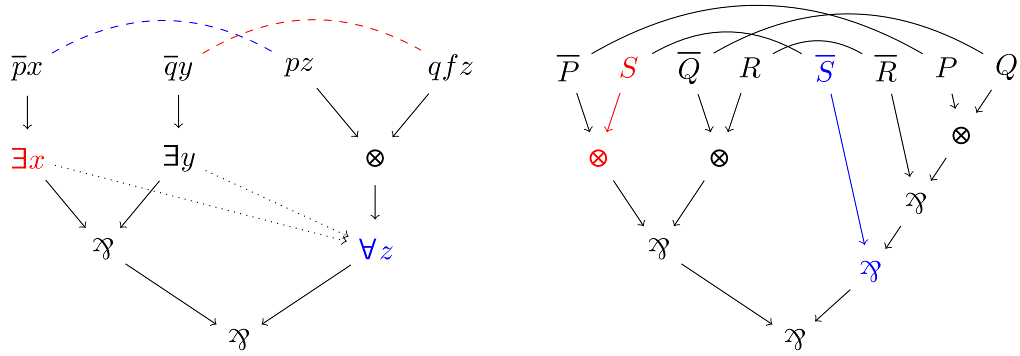

Figure 8. The unification net in Example 2.44 and its frame. The colored part shows how the dependency $\exists x \to \forall z$ is transformed.

We have the following results:

Let u and v be atoms or quantifiers in a unification structure θ. Then they are connected by a switching path in the unification structure if, and only if, their corresponding nodes are connected by a switching path in θ_m.

Consider now a switching graph H of a unification structure θ of ϕ.

If H contains a cycle, then the corresponding switching graph of θ_m also contains a cycle. Hence, by applying the propositional results (Theorem 7) from [5], we conclude that there exists a chordless, alternating, and elementary cycle in the bicoloured graph $(W, E_W \uplus L_W)$, which corresponds to an induced bimatching in the linked fograph. (Note that the linked fograph (cograph) corresponding to θ_m is equivalent to the one corresponding to θ.)

$\square$

3.3 Relation between weakening/contraction and skew bifibrations

We first introduce the atomic contraction rule, the medial rule, and two rules on quantifiers.

$$\frac{\vdash S\{a \vee a\}}{\vdash S\{a\}}\,\mathsf{ac}{\downarrow} \qquad \frac{\vdash S\{(A \wedge B) \vee (C \wedge D)\}}{\vdash S\{(A \vee C) \wedge (B \vee D)\}}\,\mathsf{m} \qquad \frac{\vdash S\{\exists x A \vee \exists x B\}}{\vdash S\{\exists x (A \vee B)\}}\,\mathsf{m_1}{\downarrow} \qquad \frac{\vdash S\{\forall x A \vee \forall x B\}}{\vdash S\{\forall x (A \vee B)\}}\,\mathsf{m_2}{\downarrow}$$

Here, we also consider the equivalence generated by the associativity, commutativity of $\vee$ and the equations $\mathsf{t} \vee A \equiv \mathsf{t}$ and $\mathsf{f} \vee A \equiv A$.

Now we have the following lemma:

Lemma 3.8. The contraction rule $\mathsf{c}{\downarrow}$ is derivable for $\{\mathsf{ac}{\downarrow}, \mathsf{m}, \mathsf{m_1}{\downarrow}, \mathsf{m_2}{\downarrow}\}$.

Proof. We prove that there is always $\begin{array}{c} A \vee A \\ \Vert\{\mathsf{ac}{\downarrow},\mathsf{m},\mathsf{m_1}{\downarrow},\mathsf{m_2}{\downarrow}\} \\ A \end{array}$ by structural induction on A.

- If $A = \mathsf{t}$ or $A = \mathsf{f}$, we have $\dfrac{\vdash S\{A \vee A\}}{\vdash S\{A\}}\equiv$. (the premiss and the conclusion are equivalent)
- If $A = a$, then we have $\dfrac{\vdash S\{a \vee a\}}{\vdash S\{a\}}\,\mathsf{ac}{\downarrow}$

- If $A = A_1 \vee A_2$, then by the induction hypothesis, we have $\begin{array}{c} A_i \vee A_i \\ \| \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow\} \\ A_i \end{array}$ for $i = 1, 2$.

  Hence, we have
$$\begin{array}{c} \dfrac{\vdash S\{(A_1 \vee A_2) \vee (A_1 \vee A_2)\}}{\vdash S\{(A_1 \vee A_1) \vee (A_2 \vee A_2)\}} \equiv \\ \vdots \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow\} \\ \vdash S\{A_1 \vee (A_2 \vee A_2)\} \\ \vdots \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow\} \\ \vdash S\{A_1 \vee A_2\} \end{array}$$

- If $A = A_1 \wedge A_2$, then by the induction hypothesis, we have $\begin{array}{c} A_i \vee A_i \\ \| \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow\} \\ A_i \end{array}$ for $i = 1, 2$.

  Hence, we have
$$\begin{array}{c} \dfrac{\vdash S\{(A_1 \wedge A_2) \vee (A_1 \wedge A_2)\}}{\vdash S\{(A_1 \vee A_1) \wedge (A_2 \vee A_2)\}} \mathsf{m} \\ \vdots \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow\} \\ \vdash S\{A_1 \wedge (A_2 \vee A_2)\} \\ \vdots \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow\} \\ \vdash S\{A_1 \wedge A_2\} \end{array}$$

- If $A = \exists x A'$, then by the induction hypothesis, we have $\begin{array}{c} A' \vee A' \\ \| \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow\} \\ A' \end{array}$.

  Hence, we have
$$\begin{array}{c} \dfrac{\vdash S\{\exists x A' \vee \exists x A'\}}{\vdash S\{\exists x (A' \vee A')\}} \mathsf{m}_1\!\downarrow \\ \vdots \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow\} \\ \vdash S\{\exists x A'\} \end{array}$$

- If $A = \forall x A'$, then by the induction hypothesis, we have $\begin{array}{c} A' \vee A' \\ \| \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow\} \\ A' \end{array}$.

  Hence, we have
$$\begin{array}{c} \dfrac{\vdash S\{\forall x A' \vee \forall x A'\}}{\vdash S\{\forall x (A' \vee A')\}} \mathsf{m}_2\!\downarrow \\ \vdots \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow\} \\ \vdash S\{\forall x A'\} \end{array}$$

□

Lemma 3.9. The rules $\mathsf{m}_1\!\downarrow$ and $\mathsf{m}_2\!\downarrow$ are derivable for $\{\mathsf{w}\!\downarrow, \mathsf{c}\!\downarrow\}$.

Proof. We have:

$$\dfrac{\dfrac{\exists x A}{\exists x (A \vee \mathsf{f})} \equiv}{\exists x (A \vee B)} \mathsf{w}\!\downarrow \quad \text{and} \quad \dfrac{\dfrac{\exists x B}{\exists x (\mathsf{f} \vee B)} \equiv}{\exists x (A \vee B)} \mathsf{w}\!\downarrow$$

Thus, we have:

$$\dfrac{\begin{array}{c}\exists x A \vee \exists x B \\ \vdots \\ \exists x(A \vee B) \vee \exists x(A \vee B)\end{array}}{\exists x(A \vee B)}\ \mathsf{c}{\downarrow}$$

Similar for $\mathsf{m}_2{\downarrow}$. ☐

Now we define a propositional encoding for first-order formulas.

Definition 3.10. The propositional encoding A^P of a formula A is defined inductively by:

$$a^P = a \text{ for every atom } a$$

$$(A \vee B)^P = A^P \vee B^P \qquad (A \wedge B)^P = A^P \wedge B^P$$

$$(\forall x A)^P = U_x \vee A^P \qquad (\exists x A)^P = E_x \wedge A^P$$

where U_x and E_x are fresh nullary atoms.

Similarly, we can define the propositional encoding S^P of a context S inductively by setting $\square^P = \square$. Note that S^P is also an encoding.

We have the following facts:

Proposition 3.11. For any context S and any formula A:

- A^P is a formula containing no quantifier for any formula A.
- $G(A^P) = G(A)$ by confounding the atoms U_x, E_x with the variable x. Thus, a map $f : G(A^P) \to G(B^P)$ can be seen as a map $f : G(A) \to G(B)$.
- $(S\{A\})^P = S^P\{A^P\}$.

Proposition 3.12. Let A and B be two formulas such that $\begin{array}{c}A\\ \Vert\{\mathsf{w}{\downarrow},\mathsf{c}{\downarrow}\}\\ B\end{array}$. Then $\begin{array}{c}A^P\\ \Vert\{\mathsf{w}{\downarrow},\mathsf{c}{\downarrow}\}\\ B^P\end{array}$.

Lemma 3.13. Given two formulas A and B and a derivation $\begin{array}{c}A\\ \Delta\Vert\{\mathsf{w}{\downarrow},\mathsf{c}{\downarrow}\}\\ B\end{array}$, then there exists a skew bifibration $G(A) \to G(B)$.

Proof. By Lemma 3.8, there exists a derivation $\begin{array}{c}A\\ \Delta\Vert\{\mathsf{w}{\downarrow},\mathsf{ac}{\downarrow},\mathsf{m},\mathsf{m}_1{\downarrow},\mathsf{m}_2{\downarrow}\}\\ B\end{array}$.

For each rule from $\{\mathsf{w}{\downarrow}, \mathsf{ac}{\downarrow}, \mathsf{m}, \mathsf{m}_1{\downarrow}, \mathsf{m}_2{\downarrow}\}$, we define a map and show that it is a skew fibration.

- $\dfrac{\vdash S\{\mathsf{f}\}}{\vdash S\{A\}}\ \mathsf{w}{\downarrow}$:
  the map *wk* maps f to anything and is identity elsewhere.
- $\dfrac{\vdash S\{a \vee a\}}{\vdash S\{a\}}\ \mathsf{ac}{\downarrow}$:
  the map *ac* maps the two a-labelled literals in the premise to the a-labelled literal in the conclusion.

- $$\frac{\vdash S\{(A \wedge B) \vee (C \wedge D)\}}{\vdash S\{(A \vee C) \wedge (B \vee D)\}}\ \mathsf{m}:$$
  the map m is the canonical identity that maps A to A, $\cdots$, D to D.
- $$\frac{\vdash S\{\exists x A \vee \exists x B\}}{\vdash S\{\exists x(A \vee B)\}}\ \mathsf{m}_1\!\downarrow:$$
  the map m_1 maps the two x-labelled binders in the premise to the x-labelled binder in the conclusion, A to A and B to B.
- $$\frac{\vdash S\{\forall x A \vee \forall x B\}}{\vdash S\{\forall x(A \vee B)\}}\ \mathsf{m}_2\!\downarrow:$$
  the map m_2 maps the two x-labelled binders in the premise to the x-labelled binder in the conclusion, A to A and B to B.

By considering propositional encodings, the maps defined are label-preserving skew fibrations on the underlying fographs according to [7].

Now we prove that each map $g \in \{wk, ac, m, m_1, m_2\}$ is a skew bifibration. To do that, it suffices to prove that g is a fibration between the corresponding binding graphs since it is already a skew fibration on the corresponding fographs and it is label-preserving and existential-preserving.

for each x-binder b in $G(B^P)$, for each vertex $v \in V(G(A^P))$ such that $g(v)$ is bound by b, there exists a unique binder b' such that b' binds v.

- wk and m are clearly fibrations: the binding relations of the premise and the conclusion are exactly the same.
- ac is a fibration: suppose that a that in the conclusion a is bound by some quantifier b in S, then for each of its preimages by ac, there exists exactly one binder (in fact, b) in S that binds it.
- m_1 and m_2 are fibrations: in the conclusion, for every atom a in $A \vee B$ bound by the x-labelled quantifier, a has exactly one preimage and it is bound by the x-llabelled quantifier in the premise.

Therefore, all of these maps are skew bifibrations and since skew bifibrations on fographs compose (Lemma 10.32, [4]), there exists a skew bifibration from $G(A)$ to $G(B)$.

□

Theorem 3.2. *If a formula ϕ is provable in* **LK***, then it has a combinatorial proof.*

Proof. By Theorem 3.1, there exists a formula ϕ' such that there is a proof Π of ϕ' in **MLL1** and a derivation D from ϕ' to ϕ consisting of the $\mathsf{w}\!\downarrow$ and $\mathsf{c}\!\downarrow$ rules only. The proof Π corresponds to a unique unification net which is equivalent to the fonet corresponding to Π, i.e., the fograph $G(\phi')$ together with the links of Π. By Lemma 3.13, there exists a skew bifibration $G(\phi') \to G(\phi)$. We have thus a combinatorial proof of ϕ.

□

4 Conclusion and Future Work

We have established a correspondence between unification nets and fonets, which allows us to prove the completeness of first-order combinatorial proofs by constructing skew bifibrations corresponding to weakening/contraction derivations. Our future goal is thus to give a proof of correctness by constructing weakening/contraction derivations from skew bifibrations given.

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