Non-Wellfounded Derivations for Intersection Subtyping with Fixpoints

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Outline

Intersection type systems and BCD subtyping

Sequent calculus IS for BCD subtyping

Extending IS with fixpoints

Instances

Conclusion and future directions

Intersection types

Intersection types are introduced to extend simple types of the λ-calculus.

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- allow typing more terms and characterizing various *qualitative* properties of reduction, such as strong normalization;
- yield models of the λ-calculus called *filter models*;
- provide a way to study *quantitative* properties of reduction, such as the number of reduction steps $\hookrightarrow$ non-idempotent intersection types.

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In this talk, $\cap$ is idempotent, that is, $A \cap A \sim A$.

Intersection type systems: typing

We consider *extensions* of the **BCD intersection type system** [1983].

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These extensions share the same set of typing rules but differ in the **subtyping** relation $\leq$ that parameterized them.

$$\frac{}{\Gamma, x : A \vdash x : A}\ var \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ abs$$

$$\frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B}\ app$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash t : B}{\Gamma \vdash t : A \cap B}\ inter \qquad \frac{}{\Gamma \vdash t : \Omega}\ omg$$

$$\frac{\Gamma \vdash t : A \qquad A \leq B}{\Gamma \vdash t : B}\ sub$$

Intersection type systems: subtyping

The subtyping relation includes the following rules:

$$\frac{}{A \leq A} \qquad \frac{A \leq B \qquad B \leq C}{A \leq C} \qquad \frac{}{A \leq \Omega}$$

$$\frac{}{A \cap B \leq A} \qquad \frac{}{A \cap B \leq B} \qquad \frac{}{A \leq A \cap A} \qquad \frac{A \leq C \qquad B \leq D}{A \cap B \leq C \cap D}$$

$$\frac{C \leq A \qquad B \leq D}{A \to B \leq C \to D} \quad \frac{}{(C \to A) \cap (C \to B) \leq C \to (A \cap B)} \quad \frac{}{\Omega \leq \Omega \to \Omega}$$

We write $A \sim B$ if $A \leq B$ and $B \leq A$.

Properties of intersection type systems

It is known from the literature that we always have:

- **subject β-expansion**: if $t \to_\beta u$ then $\Gamma \vdash u : A \Rightarrow \Gamma \vdash t : A$.
- **subject η-reduction**: if $t \to_\eta u$ then $\Gamma \vdash t : A \Rightarrow \Gamma \vdash u : A$.

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These properties require additional assumptions.

β-condition η-condition

The β-**condition**, stated as follows,

$$\bigcap_{i \in I} A_i \to B_i \leq A \to B \quad \Longrightarrow \quad \exists J \subseteq I, \quad A \leq \bigcap_{j \in J} A_j \quad \wedge \quad \bigcap_{j \in J} B_j \leq B \quad (\beta)$$

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However, checking the β-condition is still complicated...

(ii) By induction on the definition of $\leq$ one can show for $n, n', m, m' \geq 0$ that for all $l \in \{1, \ldots, n'\}$ one has

$$\begin{aligned}[&(\mu_1 \to \nu_1) \cap \cdots \cap (\mu_n \to \nu_n) \cap \varphi_{j_1} \cap \cdots \cap \varphi_{j_m} \\ &\leq (\sigma_1 \to \tau_1) \cap \cdots \cap (\sigma_{n'} \to \tau_{n'}) \cap \varphi_{j'_1} \cap \cdots \cap \varphi_{j'_{m'}} \cap \omega \cap \cdots \cap \omega, \\ &\text{and } \tau_l \not\sim \omega \Rightarrow \exists i_1, \ldots, i_k \in \{1, \ldots, n\}\ \mu_{i_1} \cap \cdots \cap \mu_{i_k} \geq \sigma_l \\ &\text{and } \nu_{i_1} \cap \cdots \cap \nu_{i_k} \leq \tau_l].\end{aligned}$$

η-condition and extensions of BCD subtyping

The η-**condition**, stated as follows,

$$\forall X \in \mathcal{A}, \quad \exists (A_i)_{i \in I} (B_i)_{i \in I}, \quad X \sim \bigcap_{i \in I} A_i \to B_i \tag{η}$$

is equivalent to subject η-expansion.

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Here are some systems from the literature we will be able to address:

Name	Atoms		Additional axioms	β	η
BCD	$\mathcal{A}$			✓	
Scott	$\mathcal{A}$		$X \sim \Omega \to X$	✓	✓
Park	$\mathcal{A}$		$X \sim X \to X$	✓	✓
CDZ	$\mathbb{B}$	$\varphi \leq \psi$	$\varphi \sim \psi \to \varphi \quad \psi \sim \varphi \to \psi$	✓	✓
HR	$\mathbb{B}$	$\varphi \leq \psi$	$\varphi \sim \psi \to \varphi \quad \psi \sim (\varphi \to \varphi) \cap (\psi \to \psi)$	✓	✓
DHM	$\mathbb{B}$	$\varphi \leq \psi$	$\varphi \sim \Omega \to \varphi \quad \psi \sim \varphi \to \psi$	✓	✓
TLCA	$\mathbb{B}$		$\varphi \sim \psi \to \varphi \quad \psi \sim (\psi \to \psi) \cap (\varphi \to \psi)$	✓	✓

where $\mathbb{B} = \{\varphi, \psi\}$.

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In 1989, Pierce proposed an algorithm for BCD subtyping, which can actually be presented as a sequent-style system.

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$$\frac{}{X \trianglelefteq \ \vdash X}\ ax \qquad \frac{}{C \trianglelefteq \Gamma \vdash \Omega}\ \Omega$$

$$\frac{C \trianglelefteq \Gamma \vdash A \qquad C \trianglelefteq \Gamma \vdash B}{C \trianglelefteq \Gamma \vdash A \cap B}\ \cap R$$

$$\frac{A \trianglelefteq \Gamma \vdash C}{A \cap B \trianglelefteq \Gamma \vdash C}\ \cap L_1 \qquad \frac{B \trianglelefteq \Gamma \vdash C}{A \cap B \trianglelefteq \Gamma \vdash C}\ \cap L_2$$

$$\frac{C \trianglelefteq \Gamma, A \vdash B}{C \trianglelefteq \Gamma \vdash A \to B}\ {\to} R \qquad \frac{C \trianglelefteq \ \vdash A \qquad B \trianglelefteq \Gamma \vdash D}{A \to B \trianglelefteq C, \Gamma \vdash D}\ {\to} L$$

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$$\frac{A, \Gamma \vdash C}{A \& B, \Gamma \vdash C}\ \& L_1 \qquad \frac{B, \Gamma \vdash C}{A \& B, \Gamma \vdash C}\ \& L_2$$

$$\frac{C, \Gamma, A \vdash B}{C, \Gamma \vdash A \multimap B}\ \multimap R \qquad \frac{C \vdash A \qquad B, \Gamma \vdash D}{A \multimap B, C, \Gamma \vdash D}\ \multimap L$$

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Cuts admissibility and equivalence

Theorem (Cuts admissibility)

The following cut rules are admissible in IS*:*

$$\frac{A \trianglelefteq \Gamma \vdash B \qquad B \trianglelefteq \Delta \vdash C}{A \trianglelefteq \Gamma, \Delta \vdash C}\ tcut \qquad \frac{A \trianglelefteq\ \vdash B \qquad C \trianglelefteq \Gamma, B, \Delta \vdash D}{C \trianglelefteq \Gamma, A, \Delta \vdash D}\ scut$$

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Theorem (Equivalence between BCD and IS)

We have $A \leq B$ *in* BCD *if and only if* $A \trianglelefteq\ \vdash B$ *has an* IS*-derivation.*

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Theorem (Equivalence between BCD and IS)

We have $A \leq B$ *in* BCD *if and only if* $A \trianglelefteq\ \vdash B$ *has an* IS*-derivation.*

We can extend BCD and IS with a preorder $\preccurlyeq$ on atoms, by adding the following two rules, respectively. This yields the systems $\mathrm{BCD}_{\preccurlyeq}$ and $\mathrm{IS}_{\preccurlyeq}$.

$$X \prec Y\ \frac{}{X \leq Y} \qquad X \preccurlyeq Y\ \frac{}{X \trianglelefteq\ \vdash Y}\ \preccurlyeq$$

$\mathsf{BCD}^{\delta}_{\preccurlyeq}$ and the non-wellfounded system $\mathsf{IS}^{\delta}_{\preccurlyeq}$

In addition to $\preccurlyeq$, we consider a function δ from atoms to types that defines "fixpoint" equations $X \sim \delta X$.

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$\mathsf{IS}^{\delta}_{\preccurlyeq}$ extends $\mathsf{IS}_{\preccurlyeq}$ by adding the following **unfolding** rules:

$$\frac{\delta X \trianglelefteq \Gamma \vdash_{\delta} B}{X \trianglelefteq \Gamma \vdash_{\delta} B}\ \mathcal{A}L \qquad \frac{A \trianglelefteq \Gamma \vdash_{\delta} \delta X}{A \trianglelefteq \Gamma \vdash_{\delta} X}\ \mathcal{A}R$$

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and by replacing the ($\preccurlyeq$) rule with the following **checkpoint** rule:

$$X \preccurlyeq Y\ \frac{X \trianglelefteq\ \vdash_{\delta} Y}{X \trianglelefteq\ \vdash_{\delta} Y}\ CP$$

$\mathsf{IS}^{\delta}_{\preccurlyeq}$-derivations

An $\mathsf{IS}^{\delta}_{\preccurlyeq}$-**derivation** is a possibly infinite tree built with the rules of $\mathsf{IS}^{\delta}_{\preccurlyeq}$ such that there are

- infinitely many checkpoints on each infinite branch, and
- exactly one $(\mathcal{A}L)$ and exactly one $(\mathcal{A}R)$ between any two consecutive checkpoints of any branch.

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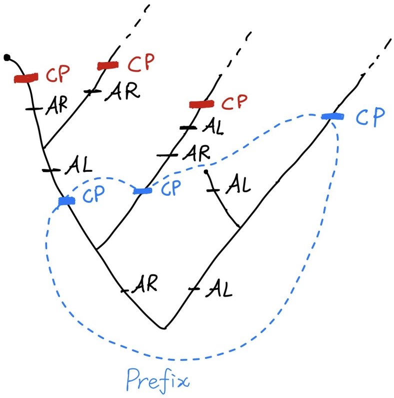

Cuts admissibility

Theorem

The following two cut rules are admissible in $\mathsf{IS}^{\delta}_{\preccurlyeq}$*:*

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Proof sketch:

We define two measures, $pw(\cdot)$ and $fw(\cdot)$, on the number of **rules** and the number of **unfolding rules** within the prefix of an $\mathsf{IS}^{\delta}_{\preccurlyeq}$-derivation.

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Proof sketch:

We define two measures, $pw(\cdot)$ and $fw(\cdot)$, on the number of **rules** and the number of **unfolding rules** within the prefix of an $\mathsf{IS}^{\delta}_{\preccurlyeq}$-derivation.

We then prove the two admissibilities by mutual induction on (f, s, p) where $f = fw(\pi_1) + fw(\pi_2)$, $s = size(B)$ and $p = pw(\pi_1) + pw(\pi_2)$.

Cut elimination: some cases

$$
\dfrac{\dfrac{C \trianglelefteq \Gamma, A \vdash_\delta B}{C \trianglelefteq \Gamma \vdash_\delta A \to B}\,{\to}R \qquad \dfrac{D \trianglelefteq\ \vdash_\delta A \quad B \trianglelefteq \Delta \vdash_\delta E}{A \to B \trianglelefteq D, \Delta \vdash_\delta E}\,{\to}L}{C \trianglelefteq \Gamma, D, \Delta \vdash_\delta E}\,\mathit{tcut}
\quad\rightsquigarrow\quad
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\quad\rightsquigarrow\quad
\dfrac{A \trianglelefteq \Gamma \vdash_\delta \delta X \qquad \delta X \trianglelefteq \Delta \vdash_\delta B}{A \trianglelefteq \Gamma, \Delta \vdash_\delta B}\,\mathit{tcut}
$$

$$
\dfrac{X \preccurlyeq Y\ \dfrac{X \trianglelefteq\ \vdash_\delta Y}{X \trianglelefteq\ \vdash_\delta Y}\,\mathit{CP} \qquad Y \preccurlyeq Z\ \dfrac{Y \trianglelefteq\ \vdash_\delta Z}{Y \trianglelefteq\ \vdash_\delta Z}\,\mathit{CP}}{X \trianglelefteq\ \vdash_\delta Z}\,\mathit{tcut}
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$$

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\dfrac{\dfrac{A \trianglelefteq \Gamma \vdash_\delta \delta X}{A \trianglelefteq \Gamma \vdash_\delta X}\,\mathcal{A}R \qquad X \preccurlyeq Y\ \dfrac{X \trianglelefteq\ \vdash_\delta Y}{X \trianglelefteq\ \vdash_\delta Y}\,\mathit{CP}}{A \trianglelefteq \Gamma \vdash_\delta Y}\,\mathit{tcut}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{A \trianglelefteq \Gamma \vdash_\delta \delta X \qquad \delta X \trianglelefteq\ \vdash_\delta \delta Y}{A \trianglelefteq \Gamma \vdash_\delta \delta Y}\,\mathit{tcut}}{A \trianglelefteq \Gamma \vdash_\delta Y}\,\mathcal{A}R
$$

Equivalence between $\mathsf{BCD}^{\delta}_{\preccurlyeq}$ and $\mathsf{IS}^{\delta}_{\preccurlyeq}$

The equivalence between $\mathsf{BCD}^{\delta}_{\preccurlyeq}$ and $\mathsf{IS}^{\delta}_{\preccurlyeq}$ requires more assumptions.

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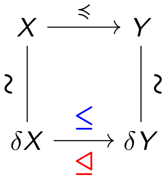
The equivalence between $\mathsf{BCD}^{\delta}_{\preccurlyeq}$ and $\mathsf{IS}^{\delta}_{\preccurlyeq}$ requires more assumptions.

We say that $(\preccurlyeq, \delta)$ is **safe** if for all X, Y such that $X \preccurlyeq Y$, we have

- $\delta X \leq \delta Y$ in $\mathsf{BCD}_{\preccurlyeq}$, or
- an $\mathsf{IS}_{\preccurlyeq}$-derivation $\tau_{X,Y}$ of $\delta X \trianglelefteq\ \vdash \delta Y$.



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$$\begin{array}{ccc} X & \xrightarrow{\preccurlyeq} & Y \\ \wr\,\Big| & & \Big|\,\wr \\ \delta X & \underset{\trianglelefteq}{\xrightarrow{\leq}} & \delta Y \end{array}$$

We say that δ is η-**safe** if δX is an intersection of arrow types for all X.

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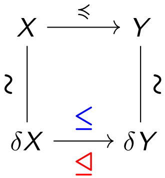

We say that δ is η-**safe** if δX is an intersection of arrow types for all X.

In the following, we assume that $(\preccurlyeq, \delta)$ is *safe*.

Equivalence between $\mathsf{BCD}^{\delta}_{\preccurlyeq}$ and $\mathsf{IS}^{\delta}_{\preccurlyeq}$

Thanks to the safety of $(\preccurlyeq, \delta)$, an $\mathsf{IS}_{\preccurlyeq}$-derivation ρ can be mapped into an $\mathsf{IS}^{\delta}_{\preccurlyeq}$-derivation $\overline{\rho}$. Consider the $(\preccurlyeq)$ leaves of ρ:

$$X \preccurlyeq Y \;\; \dfrac{}{X \trianglelefteq\; \vdash Y}\preccurlyeq \qquad \mapsto \qquad X \preccurlyeq Y \;\; \dfrac{\dfrac{\dfrac{}{\tau_{X,Y}}}{\delta X \trianglelefteq\; \vdash_\delta \delta Y} \; \mathcal{A}R, \mathcal{A}L}{\dfrac{X \trianglelefteq\; \vdash_\delta Y}{X \trianglelefteq\; \vdash_\delta Y}} \; CP$$

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In particular, forall X and Y such that $X \preccurlyeq Y$, there is an $\mathsf{IS}^{\delta}_{\preccurlyeq}$-derivation $\pi_{X,Y}$ of $X \trianglelefteq \; \vdash_{\delta} Y$.

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In particular, forall X and Y such that $X \preccurlyeq Y$, there is an $\mathsf{IS}^{\delta}_{\preccurlyeq}$-derivation $\pi_{X,Y}$ of $X \trianglelefteq\ \vdash_{\delta} Y$.

Also, forall A, there is an $\mathsf{IS}^{\delta}_{\preccurlyeq}$-derivation of $A \trianglelefteq\ \vdash_{\delta} A$.

Equivalence between $\mathsf{BCD}^{\delta}_{\lessdot}$ and $\mathsf{IS}^{\delta}_{\lessdot}$

Theorem

If we have $A \leq B$ in $\mathsf{BCD}^{\delta}_{\lessdot}$ then $A \trianglelefteq\ \vdash_{\delta} B$ has an $\mathsf{IS}^{\delta}_{\lessdot}$-derivation.

Equivalence between $\mathrm{BCD}^{\delta}_{\lessapprox}$ and $\mathrm{IS}^{\delta}_{\lessapprox}$

Theorem

If we have $A \leq B$ *in* $\mathrm{BCD}^{\delta}_{\lessapprox}$ *then* $A \trianglelefteq\ \vdash_{\delta} B$ *has an* $\mathrm{IS}^{\delta}_{\lessapprox}$*-derivation.*

Proof.

By induction on the definition of $\leq$.

□

Equivalence between $\mathrm{BCD}^{\delta}_{\preccurlyeq}$ and $\mathrm{IS}^{\delta}_{\preccurlyeq}$

Theorem

If we have $A \leq B$ in $\mathrm{BCD}^{\delta}_{\preccurlyeq}$ then $A \trianglelefteq\ \vdash_{\delta} B$ has an $\mathrm{IS}^{\delta}_{\preccurlyeq}$-derivation.

Proof.

By induction on the definition of $\leq$.

$$X \preccurlyeq Y \quad \frac{}{X \leq Y} \qquad \mapsto \qquad \pi_{X,Y}$$

$$\frac{A \leq B \qquad B \leq C}{A \leq C} \qquad \mapsto \qquad \frac{A \trianglelefteq\ \vdash_{\delta} B \qquad B \trianglelefteq\ \vdash_{\delta} C}{A \trianglelefteq\ \vdash_{\delta} C}\ tcut$$

$$\frac{}{X \leq \delta X} \qquad \mapsto \qquad \frac{\delta X \trianglelefteq\ \vdash_{\delta} \delta X}{X \trianglelefteq\ \vdash_{\delta} \delta X}\ \mathcal{A}L$$

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Equivalence between $\mathrm{BCD}^{\delta}_{\preccurlyeq}$ and $\mathrm{IS}^{\delta}_{\preccurlyeq}$

Theorem

If we have $A \leq B$ in $\mathrm{BCD}^{\delta}_{\preccurlyeq}$ then $A \trianglelefteq\ \vdash_{\delta} B$ has an $\mathrm{IS}^{\delta}_{\preccurlyeq}$-derivation.

Proof.

By induction on the definition of $\leq$.

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$$\frac{}{X \leq \delta X} \quad \mapsto \quad \frac{\delta X \trianglelefteq\ \vdash_{\delta} \delta X}{X \trianglelefteq\ \vdash_{\delta} \delta X}\ \mathcal{A}L$$

□

Theorem

If $A \trianglelefteq\ \vdash_{\delta} B$ has an $\mathrm{IS}^{\delta}_{\preccurlyeq}$-derivation then we have $A \leq B$ in $\mathrm{BCD}^{\delta}_{\preccurlyeq}$.

β- and η-conditions

Thanks to the structure of $\mathsf{IS}^{\delta}_{\preccurlyeq}$-derivations, the following lemma is easy to prove.

Lemma (Generalized β-condition for $\mathsf{IS}^{\delta}_{\preccurlyeq}$)

The following property holds in $\mathsf{IS}^{\delta}_{\preccurlyeq}$*:*

$$\bigcap_{i \in I} A_i \to B_i \trianglelefteq A, \Gamma \vdash_\delta B \quad \Longrightarrow \quad \exists J \subseteq I, \quad A \trianglelefteq \ \vdash_\delta \bigcap_{j \in J} A_j \quad \wedge \quad \bigcap_{j \in J} B_j \trianglelefteq \Gamma \vdash_\delta B$$

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Hence, $\mathsf{BCD}^{\delta}_{\preccurlyeq}$ satisfies the β-condition.

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Hence, $\mathsf{BCD}^{\delta}_{\preccurlyeq}$ satisfies the β-condition.

The η-condition is an immediate consequence of the η-safety.

Comparing with strong β systems

In the literature, one can find the following definition of strong β systems.

Definition 5 (*Strong beta preorders*). A type preorder Σ^{∇} is *strong beta* if $\nabla = \mathcal{BCD} \cup \nabla^{-}$ and:

(1) ∇^{-} contains no rule and only axioms of one of the following two shapes:

- $\psi \leqslant \psi'$,
- $\psi \sim \bigcap_{i\in I}(\psi_i^{(1)} \to \psi_i^{(2)})$,

where $\psi, \psi', \psi_i^{(1)}, \psi_i^{(2)} \in \mathbb{C}^{\nabla}$, and $\psi, \psi', \psi_i^{(2)} \not\sim \Omega$ for all $i \in I$;

(2) for each $\psi \in \mathbb{C}^{\nabla}$ such that $\psi \not\sim \Omega$ there is exactly one axiom in ∇^{-} of the shape $\psi \sim \bigcap_{i\in I}(\psi_i^{(1)} \to \psi_i^{(2)})$;

(3) let ∇^{-} contain $\psi \sim \bigcap_{i\in I}(\psi_i^{(1)} \to \psi_i^{(2)})$ and $\psi' \sim \bigcap_{j\in J}(\psi_j'^{(1)} \to \psi_j'^{(2)})$. Then ∇^{-} contains also $\psi \leqslant \psi'$ iff for each $j \in J$ there exists $i \in I$ such that $\psi_j'^{(1)} \leqslant \psi_i^{(1)}$ and $\psi_i^{(2)} \leqslant \psi_j'^{(2)}$ are both in ∇^{-}.

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Strong β systems satisfy the β- and η-conditions.

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Strong β systems satisfy the β- and η-conditions.

if $(\bigcap_{i\in I}(A_i \to B_i)) \cap (\bigcap_{h\in H} \psi_h) \leqslant_{\nabla} (\bigcap_{j\in J}(C_j \to D_j)) \cap (\bigcap_{k\in K} \varphi_k)$, then $\forall j \in J.\, (\bigcap_{i\in I'} B_i) \cap (\bigcap_{h\in H'}(\bigcap_{l\in L^{(\psi_h)'}} \zeta_l^{(\psi_h)})) \leqslant_{\nabla} D_j$ where $I' = \{i \in I \mid C_j \leqslant_{\nabla} A_i\}$, $L^{(\psi_h)'} = \{l \in L^{(\psi_h)} \mid C_j \leqslant_{\nabla} \xi_l^{(\psi_h)}\}$, $H' = \{h \in H \mid L^{(\psi_h)'} \neq \emptyset\}$;

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Proposition

If $\mathcal{S}$ is a strong β system, there exists a safe and η-safe pair $(\preccurlyeq, \delta)$ such that $\mathcal{S}$ is $\mathrm{BCD}^{\delta}_{\preccurlyeq}$.

Instances

We have already seen:

Name	Atoms	$\prec$	δ	
BCD	$\mathcal{A}$	$\varnothing$	$\delta X := X$	
Scott	$\mathcal{A}$	$\varnothing$	$\delta X := \Omega \to X$	
Park	$\mathcal{A}$	$\varnothing$	$\delta X := X \to X$	
CDZ	$\mathbb{B}$	$\varphi \prec \psi$	$\delta\varphi := \psi \to \varphi$	$\delta\psi := \varphi \to \psi$
HR	$\mathbb{B}$	$\varphi \prec \psi$	$\delta\varphi := \psi \to \varphi$	$\delta\psi := (\varphi \to \varphi) \cap (\psi \to \psi)$
DHM	$\mathbb{B}$	$\varphi \prec \psi$	$\delta\varphi := \Omega \to \varphi$	$\delta\psi := \varphi \to \psi$
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We can also consider:

Atoms	$\prec$	δ		
$\mathbb{T}$	$\varphi \prec \psi$	$\delta\varphi := \Omega \to \varphi$	$\delta\psi := \Omega \to \psi$	$\delta\kappa := \varphi \to \psi$
$\mathbb{T}$	$\varphi \prec \psi$	$\delta\varphi := \kappa \to \varphi$	$\delta\psi := \kappa \to \psi$	$\delta\kappa := \kappa \to \kappa$
$\mathbb{B}$	$\varphi \prec \psi$	$\delta\varphi := (\varphi \to \varphi \to \varphi) \cap (\varphi \to \varphi \to \psi)$	$\delta\psi := \varphi \to \varphi \to (\varphi \cap \psi)$	
$\mathbb{T}$	$\varphi \prec \psi$	$\delta\varphi := (\varphi \to \varphi \to \kappa) \cap (\varphi \to \psi \to \psi)$	$\delta\psi := \varphi \to \varphi \to (\kappa \cap \psi)$	
		$\delta\kappa := \kappa \to \kappa$		
$\vdots$	$\vdots$	$\vdots$		

where $\mathbb{T} = \{\varphi, \psi, \kappa\}$.

Conclusion and future directions

We have used techniques from proof theory to build transitivity-free presentations of a broad class of intersection subtyping systems with fixpoint equations. In particular, we generalize the "strong beta condition" to give a generic proof of the β-condition for subtyping.

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Thank you for your listening!