Proofs as Terms and Terms as Programs, Positively

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LIX, Ecole Polytechnique & Inria Saclay

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Outline

Introduction

Syntax is everywhere

We live in a world full of syntactic structures.

Terms (or expressions) are everywhere.

In programming languages, formal proofs, mathematical proofs, natural languages, etc.

Handling operations on terms can be tricky, especially with bindings.

- substitution
- equality checking
- evaluation
- sharing

Having a highly principled meta-theory is important for further studies.

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Proof theory and term representation

Need a highly prinicipled and mathematically sound meta-theory
↪ (structural) proof theory might help

Starting from a given proof system, we can obtain a term representation by annotating proofs.

In addition to the structure of terms,
other operations can sometimes be mimicked by operations on proofs.

↪ Curry-Howard correspondence.

Sometimes, operations on terms are not provided by proof theory for free, but they can be inspired by some proof-theoretic observations.

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Proofs as Terms

Gentzen's sequent calculus

We start by looking at the implicational fragment $LJ_\supset$ of Gentzen's LJ.

Formulas are made of atoms $\alpha, \beta, \ldots$, and implications $\supset$.

$$\frac{}{\Gamma, B \vdash B}\ I \qquad \frac{\Gamma, B, B \vdash C}{\Gamma, B \vdash C}\ C \qquad \frac{\Gamma \vdash B \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\ Cut$$

$$\frac{\Gamma \vdash B_1 \qquad \Gamma, B_2 \vdash B}{\Gamma, B_1 \supset B_2 \vdash B} \supset L \qquad \frac{\Gamma, B_1 \vdash B_2}{\Gamma \vdash B_1 \supset B_2} \supset R$$

Cut-elimination: the *cut* rule is not needed in terms of provability
↪ subformula property

Problems with sequent calculus

There are, however, some problems with this proof system:

1. Non-controlled contraction. Consider the proof

$$\dfrac{\begin{array}{c}\Pi \\ \Gamma, B, B \vdash C\end{array}}{\Gamma, B \vdash C}\; C$$

   Is this contraction really needed?

2. Lack of canonicity. Consider the following proofs:

$$\dfrac{}{B_1 \supset B_2 \vdash B_1 \supset B_2}\; I \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{}{B_1 \vdash B_1}\; I \qquad \dfrac{}{B_1, B_2 \vdash B_2}\; I}{B_1 \supset B_2, B_1 \vdash B_2}\; \supset L}{B_1 \supset B_2 \vdash B_1 \supset B_2}\; \supset R$$

   Are they equivalent?

These two problems become even more visible when one considers proof search.

"Flexibility in proof construction = Non-determinism in proof search"

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Solutions to the problems

1. Controlled contraction: a contraction should be directly followed by a corresponding introduction rule.

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2. Atomic initial rule & invertibility of $\supset R$.
   - We only need the initial rule for atoms: $\dfrac{}{\Gamma, \alpha \vdash \alpha} I_{at}$
   - The rule

     $$\dfrac{\Gamma, B_1 \vdash B_2}{\Gamma \vdash B_1 \supset B_2} \supset R$$

     is invertible: when doing proof-search, we can always apply $\supset R$ without losing provability.

These considerations eventually led us to the focused proof system $LJF_{\supset}$.

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Focusing, intuitively

Focusing was first introduced by Jean-Marc Andreoli as a technique to improve proof search in linear logic.

The idea is to classify inference rules based on the notion of invertibility.

The notion of invertibility provides a proof-search heuristic: whenever an invertible rule is available, one can simply apply it!

This eventually leads to focusing:

- some invertible rules are available → apply them (negative phase)
- only non-invertible rules are available → choose one formula and focus on it (and its subformulas) (positive phase)

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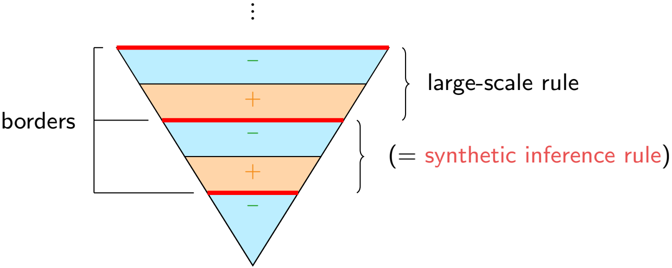
Structure of focused proofs

Proofs obtained by focusing (also called focused proofs) are cut-free and have an alternating phase structure:



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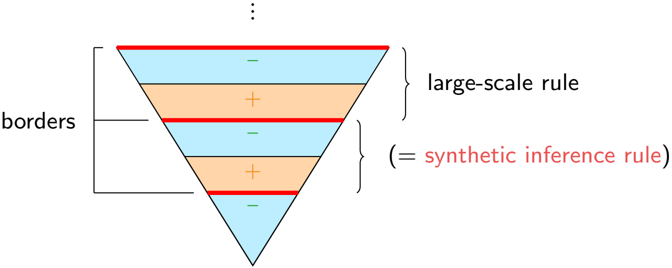

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Focused proof system $LJF_{\supset}$

In $LJF_{\supset}$, formulas are polarized: the system is equipped with a polarization $\delta : \text{ATOM} \rightarrow \{-,+\}$.

To make phases explicit, we add arrows ($\Uparrow$, $\Downarrow$) to sequents:

- Negative phase (for invertible rules) $\Gamma \vdash B \Uparrow$
- Positive phase (for non-invertible rules) $\Gamma \Downarrow B \vdash \alpha$ or $\Gamma \vdash B \Downarrow$
- Border: sequents with no arrows $\Gamma \vdash \alpha$

Rules:

$$\delta(\alpha) = - \quad \frac{}{\Gamma \Downarrow \alpha \vdash \alpha}\ I_l \qquad \delta(\alpha) = + \quad \frac{}{\Gamma, \alpha \vdash \alpha \Downarrow}\ I_R \qquad \frac{\Gamma, N \Downarrow N \vdash \alpha}{\Gamma, N \vdash \alpha}\ D_l$$

$$\delta(\alpha) = + \quad \frac{\Gamma \vdash \alpha \Downarrow}{\Gamma \vdash \alpha}\ D_r \qquad \frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \Uparrow}\ S_r \qquad \delta(\beta) = + \quad \frac{\Gamma, \beta \vdash \alpha}{\Gamma \Downarrow \beta \vdash \alpha}\ R_l \qquad \frac{\Gamma \vdash N \Uparrow}{\Gamma \vdash N \Downarrow}\ R_r$$

$$\frac{\Gamma \vdash B_1 \Downarrow \qquad \Gamma \Downarrow B_2 \vdash \alpha}{\Gamma \Downarrow B_1 \supset B_2 \vdash \alpha}\ \supset L \qquad \frac{\Gamma, B_1 \vdash B_2 \Uparrow}{\Gamma \vdash B_1 \supset B_2 \Uparrow}\ \supset R$$

Two-phase structure of $LJF_{\supset}$ proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \gamma \vdash \alpha \Downarrow}\; I_R \qquad \dfrac{}{\Gamma, \gamma \Downarrow \beta \vdash \beta}\; I_l}{\Gamma, \gamma \Downarrow \alpha \supset \beta \vdash \beta}\; \supset L}{\Gamma, \gamma \vdash \beta}\; D_l}{\Gamma, \gamma \vdash \beta \Uparrow}\; S_r}{\Gamma \Uparrow \gamma \vdash \beta \Uparrow}\; S_l}{\Gamma \vdash \gamma \supset \beta \Uparrow}\; \supset R}{\Gamma \vdash \gamma \supset \beta \Downarrow}\; R_r \qquad \dfrac{}{\Gamma \Downarrow \delta \vdash \delta}\; I_l}{\Gamma \Downarrow (\gamma \supset \beta) \supset \delta \vdash \delta}\; \supset L}{\alpha, \alpha \supset \beta, (\gamma \supset \beta) \supset \delta \vdash \delta}\; D_l$$

Synthetic inference rules in *LJF*

The synthetic inference rule for a formula B tells how B can be *used* from a proof-search point of view!

Example:
How can the formula $\alpha \supset \beta$ on the L.H.S. be used (both α and β are negative)?

$$\dfrac{\dfrac{\dfrac{\Gamma, \alpha \supset \beta \vdash \alpha}{\Gamma, \alpha \supset \beta \vdash \alpha \Uparrow}}{\Gamma, \alpha \supset \beta \vdash \alpha \Downarrow} \qquad \gamma = \beta \; \dfrac{}{\Gamma, \alpha \supset \beta \Downarrow \beta \vdash \gamma}}{\dfrac{\Gamma, \alpha \supset \beta \Downarrow \alpha \supset \beta \vdash \gamma}{\Gamma, \alpha \supset \beta \vdash \gamma}}$$

So we have the synthetic inference rule

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This depends on the polarization chosen!

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Extending $LJ_\supset$

Imagine that we want to consider some formula B to be an axiom in $LJ_\supset$.

We can simply put it on the L.H.S. However, this might not be ideal at times.

What I have:

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"Synthetic inference rules to the rescue"

synthetic

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What do proofs look like?

Let $B_1 = \alpha_0 \supset \alpha_1, \ldots, B_n = \alpha_0 \supset \cdots \supset \alpha_n, \ldots$ and consider the extensions of *LJ* by $B_1, \ldots, B_n$.

What are the proofs of $\alpha_0 \vdash \alpha_n$?

When α_i are all given the negative polarity, we have:

$$\frac{\Gamma \vdash \alpha_0}{\Gamma \vdash \alpha_1} \quad \frac{\Gamma \vdash \alpha_0 \qquad \Gamma \vdash \alpha_1}{\Gamma \vdash \alpha_2} \quad \ldots \quad \frac{\Gamma \vdash \alpha_0 \qquad \cdots \qquad \Gamma \vdash \alpha_{n-1}}{\Gamma \vdash \alpha_n} \quad \ldots$$

There is a unique proof of exponential size.

When α_i are all given the positive polarity, we have:

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Annotate rules and proofs

Now let us annotate the inference rules in the previous example.

When α_i are all given the negative polarity, we have:

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$$\frac{\Gamma \vdash \alpha_0 \qquad \cdots \qquad \Gamma \vdash \alpha_{n-1}}{\Gamma \vdash \alpha_n}$$

The unique proof of $\alpha_0 \vdash \alpha_4$ is annotated by the term:

$$\begin{gathered} B_4\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0)) \\ (B_3\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0))) \end{gathered}$$

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The shortest proof of $\alpha_0 \vdash \alpha_4$ is annotated by the term:

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Annotate rules and proofs

Now let us annotate the inference rules in the previous example.

When α_i are all given the positive polarity, we have:

$$\frac{\Gamma, x_0 : \alpha_0, x_1 : \alpha_1 \vdash t : \alpha}{\Gamma, x_0 : \alpha_0 \vdash B_1 x_0 (\lambda x_1 . t) : \alpha} \quad \frac{\Gamma, x_0 : \alpha_0, x_1 : \alpha_1, x_2 : \alpha_2 \vdash t : \alpha}{\Gamma, x_0 : \alpha_0, x_1 : \alpha_1 \vdash B_2 x_0 x_1 (\lambda x_2 . t) : \alpha} \quad \cdots$$

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\end{array}$$

Two presentations of untyped λ-terms

By considering the two axioms $\alpha \supset \alpha \supset \alpha$ and $(\alpha \supset \alpha) \supset \alpha$, and by annotating the rules, we obtain two different presentations of untyped λ-terms by using the two uniform polarizations.

α is negative

$$\alpha \in \Gamma \quad \frac{}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma \vdash \alpha \qquad \Gamma \vdash \alpha}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

negative λ-terms

α is positive

$$\alpha \in \Gamma \quad \frac{}{\Gamma \vdash \alpha}$$

$$\{\alpha, \alpha\} \sqsubseteq \Gamma \quad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

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positive λ-terms

Unfolding and substitution

In *LJF*, we have a systematic way of transforming a positively polarized proof into a negatively polarized one.

This provides a way to turn a positive λ-term into its corresponding (negative) λ-term, which consists of unfolding all the shared structures in the positive λ-term:

$$\underline{x} = x \qquad \underline{t[x\leftarrow yz]} = \underline{t}\{x\leftarrow yz\} \qquad \underline{t[x\leftarrow \lambda y.u]} = \underline{t}\{x\leftarrow \lambda y.\underline{u}\}$$

Terms correspond to cut-free proofs.

Introducing a cut between two cut-free proofs and applying cut-elimination provides a natural definition of (meta-level) substitution. For positive λ-terms,

$$t[x/ \underbrace{E\langle y\rangle}_{y[\cdots\leftarrow\cdots]\cdots[\cdots\leftarrow\cdots]}] = E\langle t\{x\leftarrow y\}\rangle$$

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Terms as Programs

λ-terms with sharing

Positive λ-terms are λ-terms with sharing.

λ-terms are given by:

$$t, u ::= x \mid tu \mid \lambda x.t$$

In CbV, there are many possible ways to restrict the shape of applications:

$$\begin{tikzcd}
 & (x\llbracket u\rrbracket)[x\leftarrow\llbracket t\rrbracket] \\
tu \arrow[ur, bend left, "\llbracket\cdot\rrbracket"] \arrow[r, "\llbracket\cdot\rrbracket"] \arrow[dr, bend right, "\llbracket\cdot\rrbracket"'] & (\llbracket t\rrbracket x)[x\leftarrow\llbracket u\rrbracket] \\
 & (xy)[x\leftarrow\llbracket t\rrbracket][y\leftarrow\llbracket u\rrbracket]
\end{tikzcd}$$

value as the left subterm of an application

value as the right subterm ⋯

values as both subterms ⋯

These restrictions are typical in a call-by-value setting, as substitutions of applications sometimes are simply blocked by the syntax:

$$xy \xrightarrow{\times} (zw)y$$

substituting zw for x

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Classification/design of call-by-value calculi with ESs

It is actually possible to have only variables as immediate sub-terms of applications.

Now we have nine different forms of applications:

- the general form tu
- eight crumbled forms vu, xu, tv', vv', xv', ty, vy, and xy.

Some more ways to classify/design call-by-value calculi with ESs.

- Nested or flattened ESs: $t[x{\leftarrow}u[y{\leftarrow}r]]$ vs. $t[x{\leftarrow}u][y{\leftarrow}r]$
- Small-step vs. micro-step substitutions:

$$(xx)[x{\leftarrow}\mathsf{I}] \to \mathsf{I}\mathsf{I}$$

$$\text{vs.}$$

$$(xx)[x{\leftarrow}\mathsf{I}] \to (\mathsf{I}x)[x{\leftarrow}\mathsf{I}] \to (\mathsf{I}\mathsf{I})[x{\leftarrow}\mathsf{I}] \to \mathsf{I}\mathsf{I}$$

- Variables as values?

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- Variables as values?

Useful substitutions v.s. non-useful substitutions

In micro-step settings, one has the following substitution rule:

$$C\langle x\rangle[x\leftarrow v] \to C\langle v\rangle[x\leftarrow v]$$

What about making a substitution only when it contributes to the creation of some β-redexes?

Consider

$$(yx)[x\leftarrow \mathsf{I}] \to (y\mathsf{I})[x\leftarrow \mathsf{I}]$$

There is no β-redex created after this substitution, and there won't be any β-redex created in the future → non-useful

Some more examples:

- $(xy)[x\leftarrow \mathsf{I}] \to (\mathsf{I}y)[x\leftarrow \mathsf{I}]$ is useful
- $x[x\leftarrow \mathsf{I}] \to \mathsf{I}[x\leftarrow \mathsf{I}]$ is non-useful

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Some more examples:

- $(xy)[x{\leftarrow}\mathsf{I}] \to (\mathsf{I}y)[x{\leftarrow}\mathsf{I}]$ is (directly) useful
- $x[x{\leftarrow}\mathsf{I}] \to \mathsf{I}[x{\leftarrow}\mathsf{I}]$ is non-useful

Usefulness: subtleties

- Contextual closure:
  $x[x{\leftarrow}\mathsf{I}] \to \mathsf{I}[x{\leftarrow}\mathsf{I}]$ is non-useful
  while $x[x{\leftarrow}\mathsf{I}]y \to \mathsf{I}[x{\leftarrow}\mathsf{I}]y$ is useful
- Indirect usefulness:
  $(xy)[x{\leftarrow}z][z{\leftarrow}\mathsf{I}] \to (xy)[x{\leftarrow}\mathsf{I}][z{\leftarrow}\mathsf{I}]$
  $\hookrightarrow$ It is useful!
- Renaming chains:

$$\begin{aligned} & (x_0 t)[x_0{\leftarrow}x_1][x_1{\leftarrow}x_2]\cdots[x_{k-1}{\leftarrow}x_k][x_k{\leftarrow}\mathsf{I}] \\ \to\quad & (x_0 t)[x_0{\leftarrow}x_1][x_1{\leftarrow}x_2]\cdots[x_{k-1}{\leftarrow}\mathsf{I}][x_k{\leftarrow}\mathsf{I}] \\ \to^*\quad & (x_0 t)[x_0{\leftarrow}\mathsf{I}][x_1{\leftarrow}\mathsf{I}]\cdots[x_{k-1}{\leftarrow}\mathsf{I}][x_k{\leftarrow}\mathsf{I}] \end{aligned}$$

Positive λ-calculus has no such issues!

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  $(xy)[x{\leftarrow}z][z{\leftarrow}\mathsf{I}] \to (xy)[x{\leftarrow}\mathsf{I}][z{\leftarrow}\mathsf{I}]$ is useful or not?
  ↪ It is useful!
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Positive λ-calculus λ_{pos}

$$t, u \quad ::= \quad x \mid t[x{\leftarrow}yz] \mid t[x{\leftarrow}\lambda y.u]$$

- ESs are flattened.
- Restricted form of explicit substitutions:
  1. Minimalistic application yz
  2. No ES for variables: variables are not values and renaming chains do not exist!

Example of reduction:

$$\begin{array}{rl} & x[x{\leftarrow}yy][y{\leftarrow}zz'][z{\leftarrow}\lambda w.w'[w'{\leftarrow}ww]] \\ \to_{\mathrm{oe}_+} & x[x{\leftarrow}yy][y{\leftarrow}(\lambda w.w'[w'{\leftarrow}ww])z'][z{\leftarrow}\lambda w.w'[w'{\leftarrow}ww]] \\ & x[x{\leftarrow}w_1'w_1'][w_1'{\leftarrow}z'z'][z{\leftarrow}\lambda w.w'[w'{\leftarrow}ww]] \end{array}$$

Key fact: λ_{pos} (resp. λ_{xpos}) is directly useful by definition!

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$$\begin{array}{ll} & x[x{\leftarrow}yy][y{\leftarrow}zz'][z{\leftarrow}\lambda w.w'[w'{\leftarrow}ww]] \\ \to_{\text{oe}_+} & x[x{\leftarrow}yy][y{\leftarrow}(\lambda w.w'[w'{\leftarrow}ww])z'][z{\leftarrow}\lambda w.w'[w'{\leftarrow}ww]] \\ \to_{\text{om}_+} & x[x{\leftarrow}w_1'w_1'][w_1'{\leftarrow}z'z'][z{\leftarrow}\lambda w.w'[w'{\leftarrow}ww]] \end{array}$$

Key fact: λ_{pos} (resp. λ_{xpos}) is directly useful by definition!

Positive λ-calculus λ_{pos}

$$t, u \quad ::= \quad x \mid t[x{\leftarrow}yz] \mid t[x{\leftarrow}\lambda y.u]$$

- ESs are flattened.
- Restricted form of explicit substitutions:
  1. Minimalistic application yz
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Explicit positive λ-calculus λ_{xpos}

$$t, u \quad ::= \quad x \mid t[x \leftarrow yz] \mid t[x \leftarrow \lambda y.u]$$

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Positive λ-calculus captures the essence of usefulness

$\lambda_{\texttt{ovsc}} \quad (= \text{Useful } \lambda_{\texttt{ovsc}} + \text{Non-useful})$

Positive λ-calculus captures the essence of usefulness

$\lambda_{\texttt{ovsc}}$ (= Useful $\lambda_{\texttt{ovsc}}$ + Non-useful)

$$t \xrightarrow{\hspace{6cm}}^{*} u$$

Positive λ-calculus captures the essence of usefulness

$\lambda_{\mathtt{ovsc}}$ $(=$ Useful $\lambda_{\mathtt{ovsc}}$ $+$ Non-useful$)$

t t' u

Positive λ-calculus captures the essence of usefulness

$$\lambda_{\texttt{ovsc}} \qquad (= \text{Useful } \lambda_{\texttt{ovsc}} + \text{Non-useful})$$

$$t \xrightarrow{\text{Useful } \lambda_{\texttt{ovsc}}}{}^{*} t' \xrightarrow{\text{Non-useful}}{}^{*} u$$

Positive λ-calculus captures the essence of usefulness

Termination

$\lambda_{\texttt{ovsc}}$ $(=$ Useful $\lambda_{\texttt{ovsc}}$ $+$ Non-useful$)$

Equivalence ⇢ ///

$$t \xrightarrow{\text{Useful } \lambda_{\texttt{ovsc}}}{}^{*} t' \xrightarrow{\text{Non-useful}}{}^{*} u$$

Positive λ-calculus captures the essence of usefulness

Termination
Equivalence

$\lambda_{\texttt{ovsc}}$ (= Useful $\lambda_{\texttt{ovsc}}$ + Non-useful)

$$\begin{tikzcd}
t \arrow[rr, "\text{Useful } \lambda_{\texttt{ovsc}}"] \arrow[d, dotted] & & {}^{*}\, t' \arrow[r, "\text{Non-useful}"] \arrow[d, dotted] & {}^{*}\, u \\
\llbracket t \rrbracket \arrow[rr, "\lambda_{\mathsf{oxpos}}"'] & \arrow[u, phantom, "\begin{array}{l}\text{preserves the number}\\ \text{of } m\text{-steps}\end{array}"] & {}^{*}\, \llbracket t' \rrbracket &
\end{tikzcd}$$

Positive λ-calculus captures the essence of usefulness

$\lambda_{\texttt{ovsc}}$ (= Useful $\lambda_{\texttt{ovsc}}$ + Non-useful)

Termination
Equivalence $\dashrightarrow \cong$

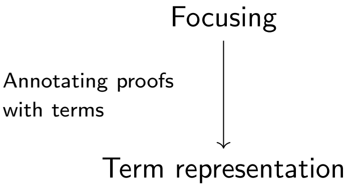
$$\begin{tikzcd}
t \arrow[r, "\text{Useful } \lambda_{\texttt{ovsc}}"] \arrow[d, dotted] \arrow[dr, phantom, "\cong"] & {}^{*}\, t' \arrow[r, "\text{Non-useful}"] \arrow[d, dotted] & {}^{*}\, u \\
{[\![t]\!]} \arrow[r, "\lambda_{\text{oxpos}}"'] & {}^{*}\, [\![t']\!]
\end{tikzcd}$$

Summing up

Focusing

Term representation

Summing up

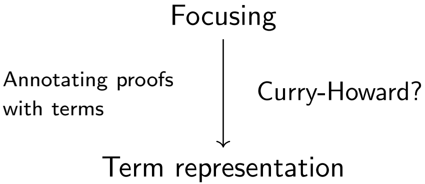


Focusing

Annotating proofs with terms

Term representation

Summing up



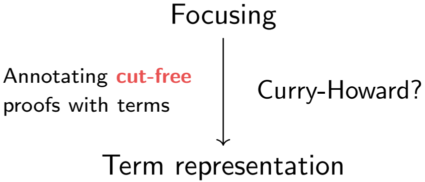
Focusing

Annotating proofs
with terms

Curry-Howard?

Term representation

Summing up



Focusing

Annotating **cut-free** proofs with terms

Curry-Howard?

Term representation

No!

Summing up

$LJF_{\supset}$

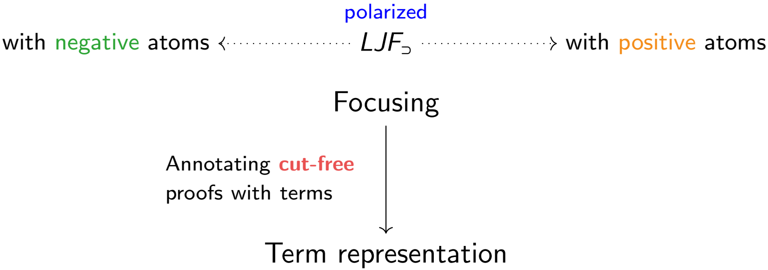
Focusing

Annotating **cut-free** proofs with terms

↓

Term representation

Summing up



polarized

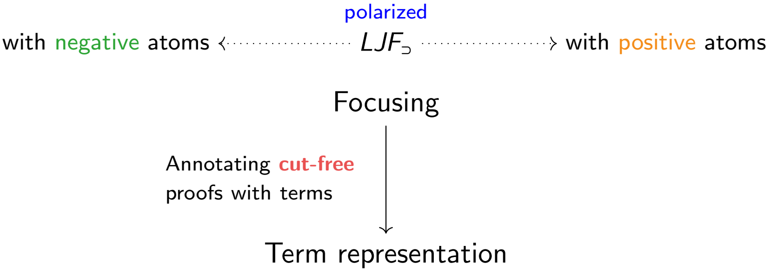
with negative atoms ⇠ $LJF_{\supset}$ ⇢ with positive atoms

Focusing

Annotating **cut-free** proofs with terms

Term representation

Summing up



polarized

with negative atoms ← $LJF_\supset$ → with positive atoms

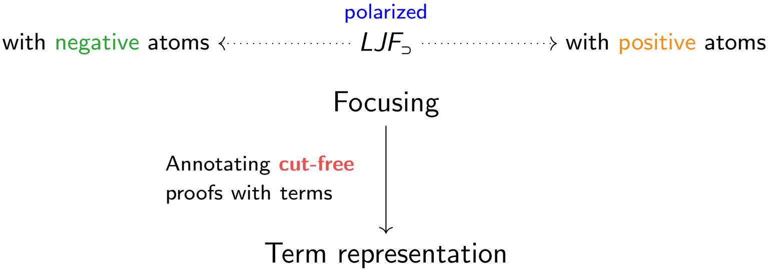
Focusing

Annotating **cut-free**
proofs with terms

Term representation

tree-like syntax
no sharing

Summing up



polarized

with negative atoms ← $LJF_{\supset}$ → with positive atoms

Focusing

Annotating **cut-free** proofs with terms

Term representation

tree-like syntax
no sharing

DAG-like syntax
allows sharing

Summing up

polarized

with negative atoms ← $LJF_{\supset}$ → with positive atoms

Focusing

↓ Annotating **cut-free** proofs with terms

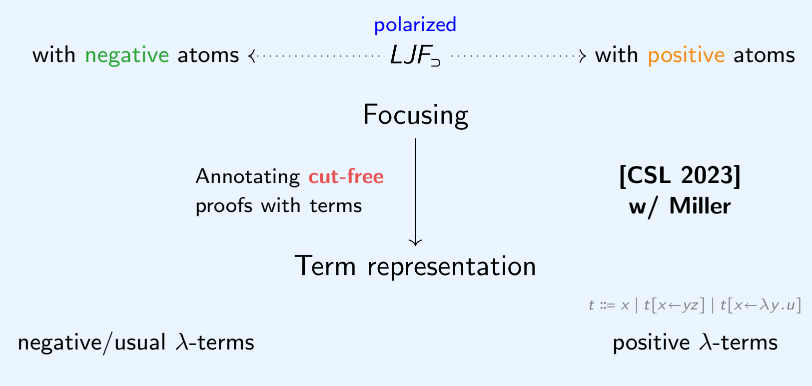
Term representation

$t ::= x \mid t[x{\leftarrow}yz] \mid t[x{\leftarrow}\lambda y.u]$

negative/usual λ-terms

positive λ-terms

Summing up



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Focusing

Annotating **cut-free** proofs with terms

Term representation

[CSL 2023] w/ Miller

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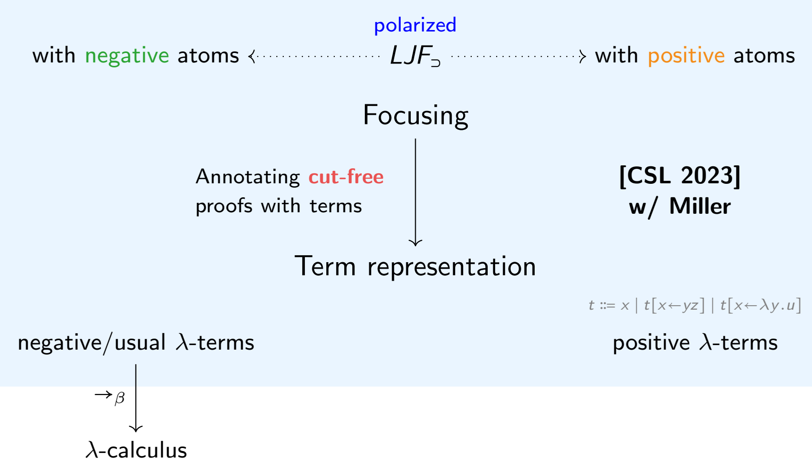
Summing up



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Focusing

Annotating **cut-free** proofs with terms

[CSL 2023] w/ Miller

Term representation

$t ::= x \mid t[x{\leftarrow}yz] \mid t[x{\leftarrow}\lambda y.u]$

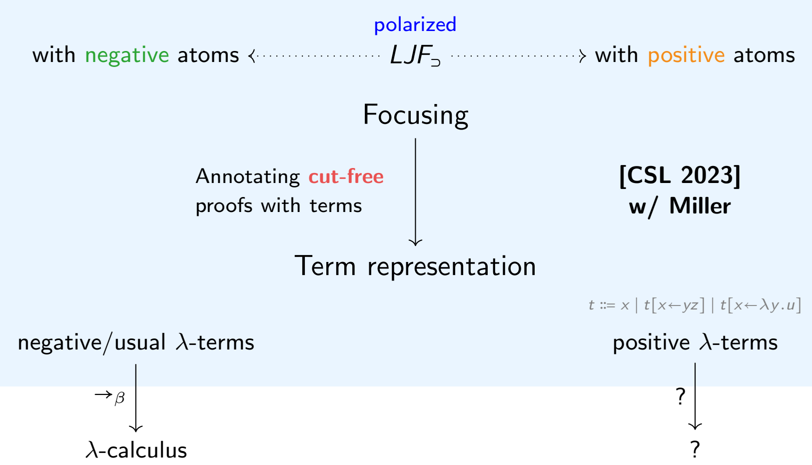
negative/usual λ-terms

positive λ-terms

$\rightarrow_{\beta}$

λ-calculus

Summing up



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Focusing

Annotating **cut-free** proofs with terms

[CSL 2023] w/ Miller

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?

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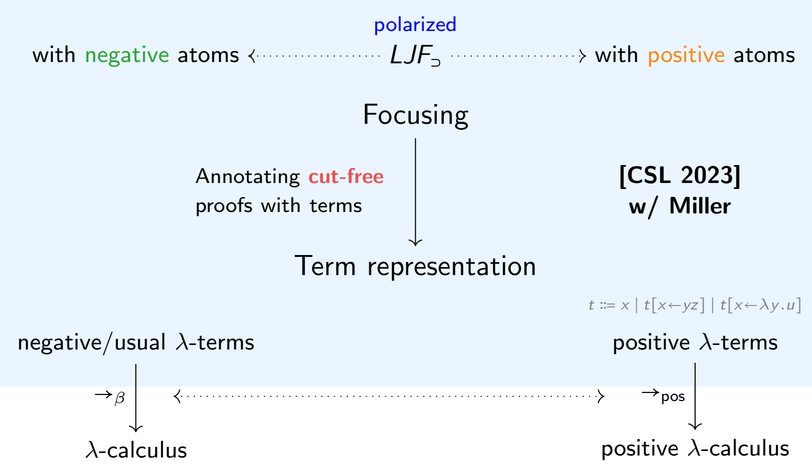
Summing up



polarized

with negative atoms ⟵ $LJF_{\supset}$ ⟶ with positive atoms

Focusing

Annotating **cut-free** proofs with terms

[CSL 2023] w/ Miller

Term representation

$t ::= x \mid t[x \leftarrow yz] \mid t[x \leftarrow \lambda y.u]$

negative/usual λ-terms

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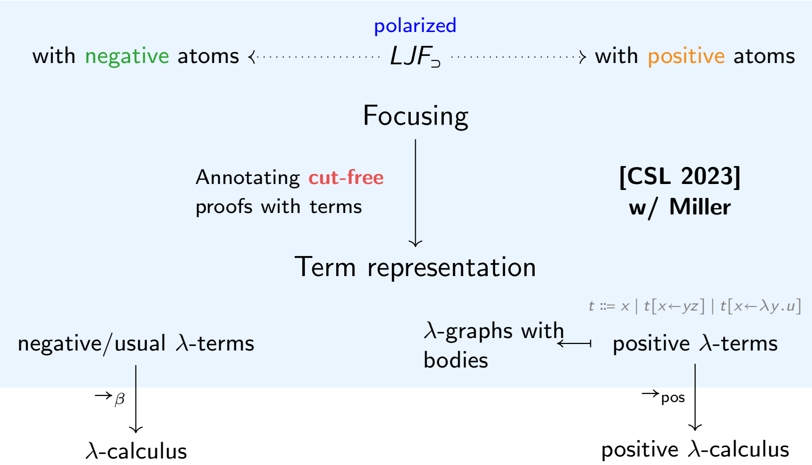
$\rightarrow_\beta$

$\rightarrow_{\text{pos}}$

λ-calculus

positive λ-calculus

Summing up



polarized

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Focusing

Annotating **cut-free** proofs with terms

[CSL 2023] w/ Miller

Term representation

$t ::= x \mid t[x \leftarrow yz] \mid t[x \leftarrow \lambda y.u]$

negative/usual λ-terms

λ-graphs with bodies ⟻ positive λ-terms

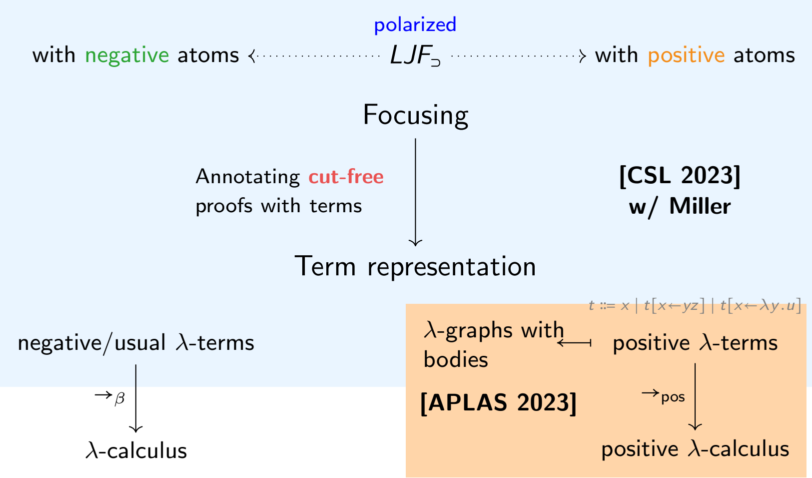
$\to_\beta$

$\to_{\text{pos}}$

λ-calculus

positive λ-calculus

Summing up



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Focusing

Annotating **cut-free** proofs with terms

Term representation

[CSL 2023] w/ Miller

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negative/usual λ-terms

$\rightarrow_{\beta}$

λ-calculus

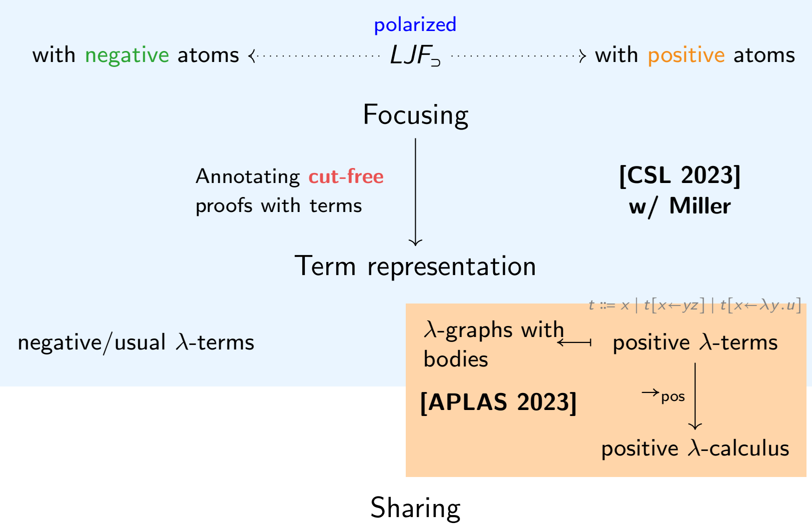
λ-graphs with bodies ⟵ positive λ-terms

[APLAS 2023]

$\rightarrow_{\text{pos}}$

positive λ-calculus

Summing up



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with negative atoms $\leftarrow$ $LJF_{\supset}$ $\rightarrow$ with positive atoms

Focusing

Annotating **cut-free** proofs with terms

[CSL 2023]
w/ Miller

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negative/usual λ-terms

λ-graphs with bodies $\longleftarrow$ positive λ-terms

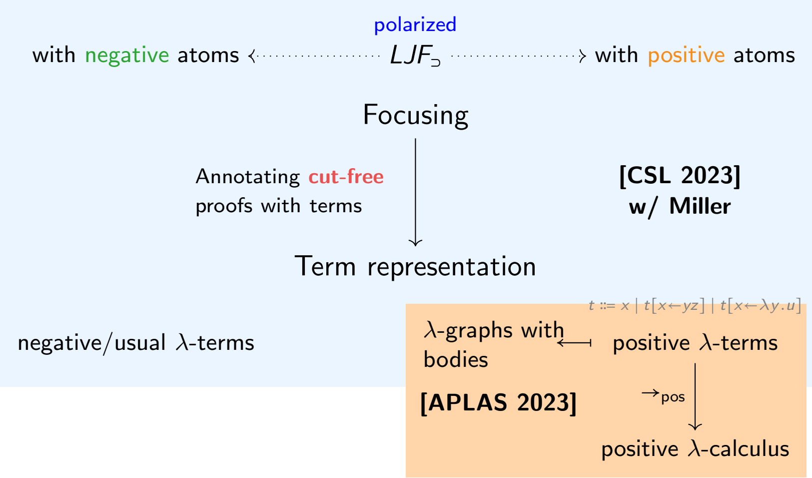
$\rightarrow_{\text{pos}}$

positive λ-calculus

[APLAS 2023]

Sharing

Summing up



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Focusing

Annotating **cut-free**
proofs with terms

[CSL 2023]
w/ Miller

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λ-graphs with bodies ⟵ positive λ-terms

[APLAS 2023]

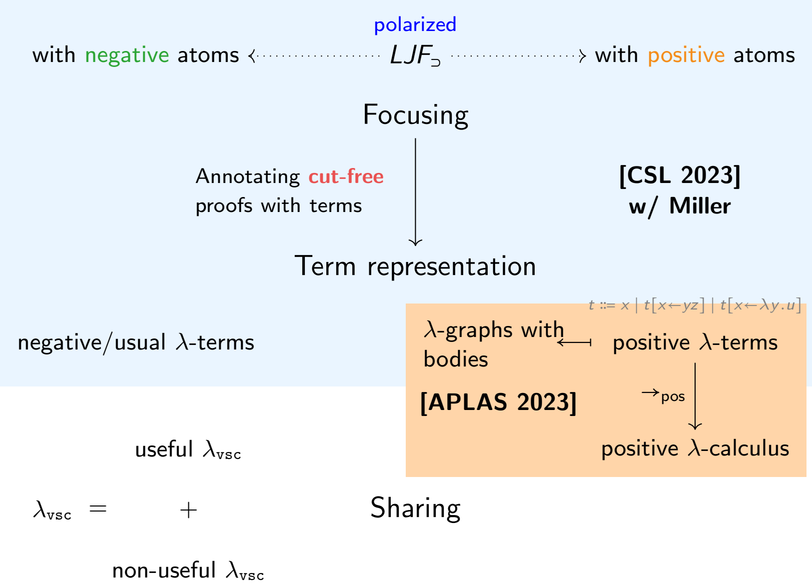
$\to_{\text{pos}}$

positive λ-calculus

λ_{vsc}

Sharing

Summing up



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Focusing

Annotating **cut-free** proofs with terms

[CSL 2023]
w/ Miller

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[APLAS 2023]

$\rightarrow_{\text{pos}}$

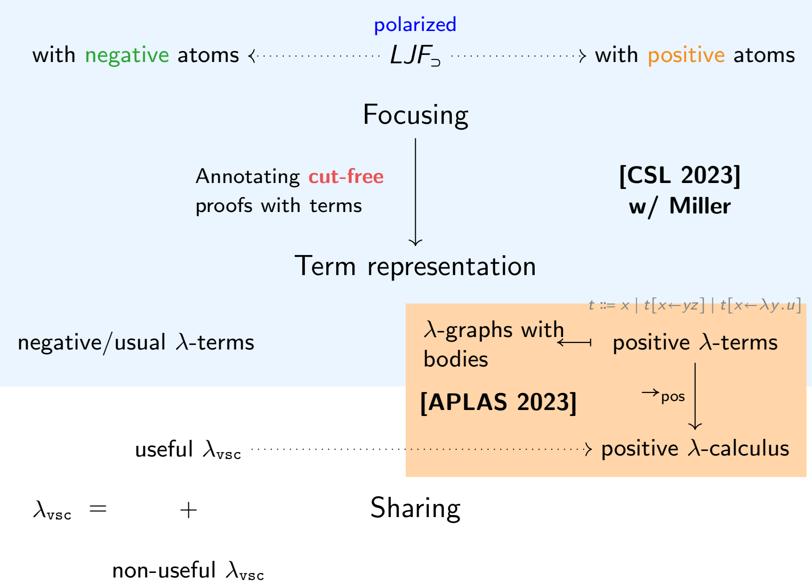
positive λ-calculus

useful λ_{vsc}

$\lambda_{\text{vsc}} = \quad + \quad$ Sharing

non-useful λ_{vsc}

Summing up



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Focusing

Annotating **cut-free** proofs with terms

[CSL 2023] w/ Miller

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negative/usual λ-terms

λ-graphs with bodies ⟵ positive λ-terms

[APLAS 2023]

$\rightarrow_{\text{pos}}$

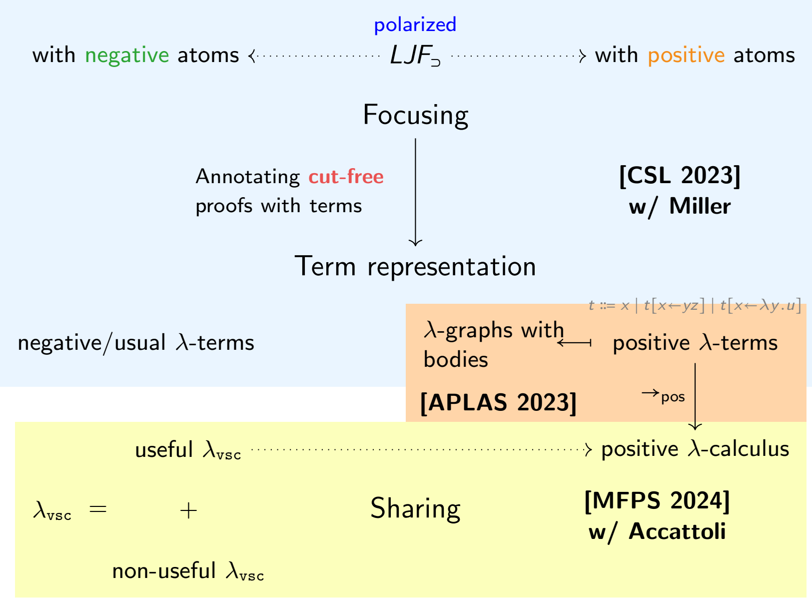
useful λ_{vsc} ⇢ positive λ-calculus

$\lambda_{\text{vsc}} \;=\; +$

Sharing

non-useful λ_{vsc}

Summing up



polarized

with negative atoms $\leftarrow$ $LJF_{\supset}$ $\rightarrow$ with positive atoms

Focusing

Annotating **cut-free**
proofs with terms

[CSL 2023]
w/ Miller

Term representation

$t ::= x \mid t[x{\leftarrow}yz] \mid t[x{\leftarrow}\lambda y.u]$

negative/usual λ-terms

λ-graphs with
bodies

positive λ-terms

[APLAS 2023]

$\to_{\text{pos}}$

useful λ_{vsc} $\rightarrow$ positive λ-calculus

λ_{vsc} $=$ $+$

Sharing

[MFPS 2024]
w/ Accattoli

non-useful λ_{vsc}

Future work

- Towards a better understanding of polarities (of atomic formulas) in full linear logic.
- Efficient implementation of meta-level renamings involved in λ_{pos}. We expect this to be doable in an efficient way via an appropriate abstract machine.
- λ_{pos} for call-by-need evaluation.

Thank you for listening!