Focusing and polarization: a journey from proof search to term representation

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Proof theory

In proof theory, we study proofs.

In structural proof theory, we study the structure of proofs.

Proof systems are used to describe how we can construct proofs for a given underlying logic (classical logic, intuitionistic logic, linear logic).

Typical proof systems include:

- Hilbert-style systems
- Natural deduction (Gentzen, Prawitz)
- Sequent calculus (Gentzen)

We can:

- find a proof of a given formula: proof search
- ask if two proofs should be considered equivalent: proof identity/proof canonicity
- ask whether a proof is *normal*: proof normalization

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Sequent calculus, briefly

Invented by Gentzen as a tool for studying natural deduction.

Some key features of sequent calculus:

- A sequent is of the form $B_1, \ldots, B_m \vdash C_1, \ldots, C_n$. Intuitively, such a sequent corresponds to the formula $(B_1 \wedge \cdots \wedge B_m) \supset (C_1 \vee \cdots \vee C_n)$
- left/right introduction rules instead of introduction/elimination rules in natural deduction → more symmetry
- Cut rule:

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

  Hauptsatz (Cut-elimination): the cut rule is not really needed.

  Cut-free proofs are analytic: a cut-free proof of a sequent contains only sub-formulas of some formula from that sequent.
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Sequent calculus proofs lack structure

Sequent calculus proofs often contain too much information.

Consider the following two proofs:

$$\dfrac{\dfrac{\dfrac{\vdots}{A_1, A_2, B_1, B_2 \vdash C}}{A_1, A_2, B_1 \wedge B_2 \vdash C}\ \wedge L}{A_1 \wedge A_2, B_1 \wedge B_2 \vdash C}\ \wedge L \qquad \dfrac{\dfrac{\dfrac{\vdots}{A_1, A_2, B_1, B_2 \vdash C}}{A_1 \wedge A_2, B_1, B_2 \vdash C}\ \wedge L}{A_1 \wedge A_2, B_1 \wedge B_2 \vdash C}\ \wedge L$$

These two proofs are "essentially" the same via rule permutation. This is due to the syntactic bureaucracy of sequent calculus.
↪ Girard's proof nets.

Another issue is the explosion of search space during backward proof search: if there are 1000 non-atomic formulas on the L.H.S., there are (at least) 1000 possible choices for the left introduction rule.
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Focusing, intuitively

Focusing was first introduced by Jean-Marc Andreoli as a technique to improve proof search in linear logic.

The idea is to classify inference rules based on the notion of invertibility.

Proofs obtained by focusing (also called focused proofs) have an alternating phase structure:

Proofs can be seen as built with some larger units rather than tiny inference rules.
↪ phases, synthetic connectives, synthetic inference rules, etc.

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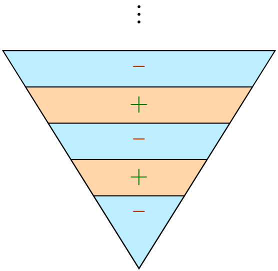

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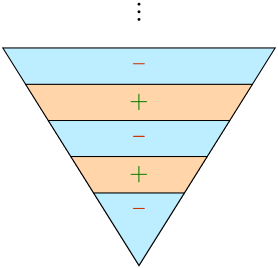

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MLL (multiplicative linear logic)

MLL formulas are given by:

$$A, B ::= \alpha \mid \alpha^{\perp} \mid A \overbrace{\parr}^{\text{disj.}} B \mid A \overbrace{\otimes}^{\text{conj.}} B$$

De morgan duals:

$$\begin{array}{rcl} (\alpha^{\perp})^{\perp} & = & \alpha \\ (A \parr B)^{\perp} & = & A^{\perp} \otimes B^{\perp} \\ (A \otimes B)^{\perp} & = & A^{\perp} \parr B^{\perp} \end{array}$$

Inference rules:

$$\frac{}{\vdash A, A^{\perp}}\ ax \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash \Gamma_1, A \quad \vdash \Gamma_2, B}{\vdash \Gamma_1, A \otimes B, \Gamma_2}\ \otimes$$

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When reading rules from conclusion to premises,

- the $\parr$ rule is deterministic, and
- the $\otimes$ rule is non-deterministic (the side context Γ_1, Γ_2 has to be *splitted*).

Let us look for a proof of $\vdash \alpha^{\perp} \otimes (\beta^{\perp} \otimes \gamma^{\perp}), (\alpha \parr \beta) \parr \gamma$.

Note that in this example,

- Applying first the $\otimes$ rule never leads to a proof.
- The unique proof begins with two $\parr$ rules.

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Invertibility of inference rules

In fact, the $\parr$ rule is invertible and the $\otimes$ rule is non-invertible.

An inference rule is called *invertible* means that if its conclusion has a proof, then all of its premises must have a proof.

The notion of invertibility provides a proof-search heuristic: whenever an invertible rule is available, one can simply apply it!

This eventually leads to focusing:

- when some invertible rules are available, apply them (in some order) $\rightsquigarrow$ negative phase
- otherwise, only non-invertible (logical) rules are available. Choose one corresponding formula and focus on it (and its s), until some invertible rules become available again $\rightsquigarrow$ positive phase

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A focused proof

$$
\begin{prooftree}
\AxiomC{$\vdash A, A^\perp$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash B, B^\perp$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash C^\perp, C$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash B, B^\perp \otimes C^\perp, C$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash A, A^\perp \otimes (B^\perp \otimes C^\perp), B, C$}
\RightLabel{$\parr$}
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\AxiomC{}
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\RightLabel{$\parr$}
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\end{prooftree}
$$

Polarities in linear logic

In linear logic, we have the following duality:

$$\text{the right introduction of a connective is invertible}$$
$$\Updownarrow$$
$$\text{the right introduction of its dual connective is non-invertible}$$

We can then define the polarity of a connective.

A connective is negative (resp. positive) if its right introduction rule is invertible (resp. non-invertible).

This definition depends on the inference rules chosen for connectives:

- no ambiguity in linear logic
- some ambiguities in intuitionistic and classical logics

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What about Gentzen's *LJ*?

A natural question arises:

Can we do the same for Gentzen's *LJ*?

With the presence of (general) structural rules, the situation is more complicated.

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ c \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ w$$

In Gentzen's *LJ*, we have the following non-invertible left introduction rules for $\wedge$:

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge L_1 \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge L_2$$

However, we can replace these rules with the following invertible rule:

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So, is $\wedge$ positive or negative? Or both?

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So, is $\wedge$ positive or negative? Or both?

Polarities and evaluation strategies

Different choices of polarizing type expressions have been related to different evaluation strategies such as call-by-name and call-by-value via the Curry-Howard correspondence.

Here are some (non-exhaustive) references:

- *LKT* and *LKQ* by Danos, Joinet, and Schellinx
- $\lambda\mu\tilde{\mu}$-calculus by Curien and Herbelin
- Dual calculus by Wadler
- System *L* by Munch-Maccagnoni

In this talk, I will not discuss this aspect and only focus on cut-free proofs.

Polarizing atomic formulas

Atomic formulas can also be polarized.

Consider the following instance of $\supset L$ rule:

$$\frac{\Gamma \vdash \alpha \qquad \Gamma, \beta \vdash A}{\Gamma, \alpha \supset \beta \vdash A} \supset L$$

One can consider two proof search protocols:

- T-protocal (T for "tête", head in French): The right branch is trivial, that is, $\beta = A$. Continue the search with $\Gamma \vdash \alpha$.
- Q-protocol (Q for "queue", tail in French): The left branch is trivial, that is, $\Gamma = \Gamma', \alpha$. Continue the search with $\Gamma', \alpha, \beta \vdash A$.

For example, *LJT* by Herbelin follows the T-protocol while *LJQ*' by Dyckhoff and Lengrand follows the Q-protocol.

As we shall see later, the T-protocol (resp. Q-protocol) is followed when atoms are polarized negatively (resp. positively).

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LJF

Liang and Miller proposed a focused proof system *LJF* implementing all these considerations.

Formulas are polarized:

$$A, B ::= \alpha \mid A \wedge^- B \mid A \wedge^+ B \mid A \vee^+ B \mid A \supset B$$

Focused sequents ($\Downarrow$-sequents) for non-invertible rules:

- $\Gamma \Downarrow A \vdash B$ with a left focus on A
- $\Gamma \vdash A \Downarrow$ with a right focus on A

Unfocused sequents ($\Uparrow$-sequents) for invertible rules:

- $\Gamma \Uparrow \Delta \vdash \Theta_1 \Uparrow \Theta_2$ with formulas in Δ and Θ_1 to be treated.

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LJF: $\Uparrow$ phase

$$\frac{\Gamma \Uparrow B_1 \vdash B_2 \Uparrow}{\Gamma \Uparrow \cdot \vdash B_1 \supset B_2 \Uparrow} \supset R \qquad \frac{\Gamma \Uparrow \cdot \vdash B_1 \Uparrow \qquad \Gamma \Uparrow \cdot \vdash B_2 \Uparrow}{\Gamma \Uparrow \cdot \vdash B_1 \wedge^- B_1 \Uparrow} \wedge^- R$$

$$\frac{\Gamma \Uparrow \Delta, B_1, B_2 \vdash \Theta_1 \Uparrow \Theta_2}{\Gamma \Uparrow \Delta, B_1 \wedge^+ B_2 \vdash \Theta_1 \Uparrow \Theta_2} \vee^+ L$$

$$\frac{\Gamma \Uparrow \Delta, B_1 \vdash \Theta_1 \Uparrow \Theta_2 \qquad \Gamma \Uparrow \Delta, B_2 \vdash \Theta_1 \Uparrow \Theta_2}{\Gamma \Uparrow \Delta, B_1 \wedge^+ B_2 \vdash \Theta_1 \Uparrow \Theta_2} \wedge^+ L$$

The $\Uparrow$ phase consists of applying invertible rules to formulas in the two middle zones. These zones are treated as lists instead of multisets as invertible rules can be applied in any order.

If one of these zones is empty, we often drop its corresponding arrow. A border sequent is of the form $\Gamma \vdash B$.

LJF: $\Uparrow$ phase

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$$\frac{\Gamma \Uparrow \Delta, B_1, B_2 \vdash \Theta_1 \Uparrow \Theta_2}{\Gamma \Uparrow \Delta, B_1 \wedge^+ B_2 \vdash \Theta_1 \Uparrow \Theta_2} \vee^+ L$$

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LJF: $\Downarrow$ phase

$$\frac{\Gamma \vdash B_1 \Downarrow \qquad \Gamma \vdash B_2 \Downarrow}{\Gamma \vdash B_1 \wedge^+ B_2 \Downarrow} \wedge^+ R \qquad \frac{\Gamma \vdash B_i \Downarrow}{\Gamma \vdash B_1 \vee^+ B_2 \Downarrow} \vee^+ R_i$$

$$\frac{\Gamma \Downarrow B_i \vdash B}{\Gamma \Downarrow B_1 \wedge^- B_2 \vdash B} \wedge^- L_i \qquad \frac{\Gamma \vdash B_1 \Downarrow \qquad \Gamma \Downarrow B_2 \vdash B}{\Gamma \Downarrow B_1 \supset B_2 \vdash B} \supset L$$

The $\Downarrow$ phase consists of applying non-invertible rule to formula under focus (and then its sub-formulas).

Note that we need two kinds of $\Downarrow$-sequents because of the $\supset L$ rule.

Identity and structural rules

Initial rules:

$$\frac{\alpha \text{ negative}}{\Gamma \Downarrow \alpha \vdash \alpha}\ I_l \qquad \frac{\alpha \text{ positive}}{\Gamma, \alpha \vdash \alpha \Downarrow}\ I_R$$

Decide rules:

$$\frac{\Gamma, N \Downarrow N \vdash B}{\Gamma, N \vdash B}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow}{\Gamma \vdash P}\ D_R$$

Release rules:

$$\frac{\Gamma \Uparrow P \vdash B}{\Gamma \Downarrow P \vdash B}\ R_l \qquad \frac{\Gamma \vdash N \Uparrow}{\Gamma \vdash N \Downarrow}\ R_r$$

Store rules:

$$\frac{\Gamma, C \Uparrow \Delta \vdash \Theta_1 \Uparrow \Theta_2}{\Gamma \Uparrow C, \Delta \vdash \Theta_1 \Uparrow \Theta_2}\ S_l \qquad \frac{\Gamma \vdash D}{\Gamma \vdash D \Uparrow}\ S_r$$

Key results of *LJF*

A polarization of an *LJ* formula B is obtained by:

- giving each atomic formula either the positive or negative polarity (two occurrences of the same atomic formula must have the same polarity),
- replacing each *occurrence* of $\wedge$ with either $\wedge^+$ or $\wedge^-$, and
- replacing $\vee$ with $\vee^+$.

Theorem (Soundness of completeness)

$\vdash B$ is provable in LJ if and only if $\vdash \tilde{B} \Uparrow$ is provable in LJF for some polarization $\tilde{B}$ of B.

Polarization does not affect provability of a sequent but can have a big impact on the structure of proofs. This feature gives us the possibility of designing different styles of term structures.

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Synthetic inference rules

Synthetic inference rule = large-scale rule = $\Downarrow$-phase + $\Uparrow$-phase

Definition

A *(left) synthetic inference rule* for a negative formula N is an inference rule of the form

$$\frac{N, \Gamma_1 \vdash \alpha_1 \qquad \ldots \qquad N, \Gamma_n \vdash \alpha_n}{N, \Gamma \vdash \alpha}\ N$$

justified by an *LJF* derivation of the form

$$\begin{array}{c} N, \Gamma_1 \vdash \alpha_1 \qquad \ldots \qquad N, \Gamma_n \vdash \alpha_n \\ \vdots\ \Uparrow\text{-phase} \\ \\ \vdots\ \Downarrow\text{-phase} \\ \dfrac{N, \Gamma \Downarrow N \vdash \alpha}{N, \Gamma \vdash \alpha}\ D_l \end{array}$$

Synthetic inference rules

First remark:
for all i, $\Gamma \subseteq \Gamma_i$ and $\Gamma_i \setminus \Gamma$ depends only on N.

A natural question arises: for which N $\Gamma_i \setminus \Gamma$ are particularly simple?

Order of a formula:

- $ord(\alpha) = 0$
- $ord(B_1 \supset B_2) = max(ord(B_1) + 1, ord(B_2))$

If $ord(N) = k$, then $ord(C) \leq k - 2$ for all $C \in \Gamma_i \setminus \Gamma$.

In particular, $\Gamma_i \setminus \Gamma$ contains only atomic formulas if $ord(N) \leq 2$.

$$
\dfrac{\dfrac{N, \Gamma_1 \vdash \alpha_1 \quad \dots \quad N, \Gamma_n \vdash \alpha_n}{\vdots \ \Uparrow\text{-phase}} \quad \vdots \ \Downarrow\text{-phase}}{}
$$

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Axioms as rules (Marin et al.)

Definition (Extensions of *LJ* by polarized theories)

Let T be a finite polarized theory of order at most 2. For every synthetic inference rule

$$\frac{N, \Gamma_1 \vdash \alpha_1 \quad \dots \quad N, \Gamma_n \vdash \alpha_n}{N, \Gamma \vdash \alpha}\ N$$

with $N \in T$, the extension $LJ\langle T\rangle$ of LJ by the polarized theory T includes the inference rule

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$\hookrightarrow$ Make axioms implicit by adding rules.

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$T, \Gamma \vdash B$ provable in LJ $\Leftrightarrow$ $\Gamma \vdash B$ provable in $LJ\langle T\rangle$.

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Polarities of atoms and forms of rules

Let T be the collection of formulas
$B_1 = \alpha_0 \supset \alpha_1, \ldots, B_n = \alpha_0 \supset \cdots \supset \alpha_n, \ldots$ where α_i are all atomic.

If α_i are all given the negative polarity, then $LJ\langle T\rangle$ includes

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What do proofs look like?

What are the proofs of $\alpha_0 \vdash \alpha_n$?

When α_i are all given the negative polarity, we have:

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There is a unique proof of exponential size.

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$$\frac{\Gamma \vdash \alpha_0}{\Gamma \vdash \alpha_1} \quad \frac{\Gamma \vdash \alpha_0 \qquad \Gamma \vdash \alpha_1}{\Gamma \vdash \alpha_2} \quad \dots \quad \frac{\Gamma \vdash \alpha_0 \qquad \cdots \qquad \Gamma \vdash \alpha_{n-1}}{\Gamma \vdash \alpha_n} \quad \dots$$

There is a unique proof of exponential size.

When α_i are all given the positive polarity, we have:

$$\frac{\Gamma, \alpha_0, \alpha_1 \vdash \alpha}{\Gamma, \alpha_0 \vdash \alpha} \quad \frac{\Gamma, \alpha_0, \alpha_1, \alpha_2 \vdash \alpha}{\Gamma, \alpha_0, \alpha_1 \vdash \alpha} \quad \dots \quad \frac{\Gamma, \alpha_0, \dots, \alpha_{n-1}, \alpha_n \vdash \alpha}{\Gamma, \alpha_0, \dots, \alpha_{n-1} \vdash \alpha} \quad \dots$$

There is a shortest proof of linear size.

What do proofs look like?

What are the proofs of $\alpha_0 \vdash \alpha_n$?

When α_i are all given the negative polarity, we have:

$$\frac{\Gamma \vdash \alpha_0}{\Gamma \vdash \alpha_1} \quad \frac{\Gamma \vdash \alpha_0 \qquad \Gamma \vdash \alpha_1}{\Gamma \vdash \alpha_2} \quad \dots \quad \frac{\Gamma \vdash \alpha_0 \qquad \cdots \qquad \Gamma \vdash \alpha_{n-1}}{\Gamma \vdash \alpha_n} \quad \dots$$

There is a unique proof of exponential size.

When α_i are all given the positive polarity, we have:

$$\frac{\Gamma, \alpha_0, \alpha_1 \vdash \alpha}{\Gamma, \alpha_0 \vdash \alpha} \quad \frac{\Gamma, \alpha_0, \alpha_1, \alpha_2 \vdash \alpha}{\Gamma, \alpha_0, \alpha_1 \vdash \alpha} \quad \dots \quad \frac{\Gamma, \alpha_0, \dots, \alpha_{n-1}, \alpha_n \vdash \alpha}{\Gamma, \alpha_0, \dots, \alpha_{n-1} \vdash \alpha} \quad \dots$$

There is a shortest proof of linear size.

What do proofs look like?

What are the proofs of $\alpha_0 \vdash \alpha_n$?

When α_i are all given the negative polarity, we have:

$$\frac{\Gamma \vdash \alpha_0}{\Gamma \vdash \alpha_1} \quad \frac{\Gamma \vdash \alpha_0 \qquad \Gamma \vdash \alpha_1}{\Gamma \vdash \alpha_2} \quad \dots \quad \frac{\Gamma \vdash \alpha_0 \qquad \cdots \qquad \Gamma \vdash \alpha_{n-1}}{\Gamma \vdash \alpha_n} \quad \dots$$

There is a unique proof of exponential size.

When α_i are all given the positive polarity, we have:

$$\frac{\Gamma, \alpha_0, \alpha_1 \vdash \alpha}{\Gamma, \alpha_0 \vdash \alpha} \quad \frac{\Gamma, \alpha_0, \alpha_1, \alpha_2 \vdash \alpha}{\Gamma, \alpha_0, \alpha_1 \vdash \alpha} \quad \dots \quad \frac{\Gamma, \alpha_0, \dots, \alpha_{n-1}, \alpha_n \vdash \alpha}{\Gamma, \alpha_0, \dots, \alpha_{n-1} \vdash \alpha} \quad \dots$$

There is a shortest proof of linear size.

What do proofs look like?

What are the proofs of $\alpha_0 \vdash \alpha_n$?

When α_i are all given the negative polarity, we have:

$$\frac{\Gamma \vdash \alpha_0}{\Gamma \vdash \alpha_1} \quad \frac{\Gamma \vdash \alpha_0 \qquad \Gamma \vdash \alpha_1}{\Gamma \vdash \alpha_2} \quad \dots \quad \frac{\Gamma \vdash \alpha_0 \qquad \cdots \qquad \Gamma \vdash \alpha_{n-1}}{\Gamma \vdash \alpha_n} \quad \dots$$

There is a unique proof of exponential size.

When α_i are all given the positive polarity, we have:

$$\frac{\Gamma, \alpha_0, \alpha_1 \vdash \alpha}{\Gamma, \alpha_0 \vdash \alpha} \quad \frac{\Gamma, \alpha_0, \alpha_1, \alpha_2 \vdash \alpha}{\Gamma, \alpha_0, \alpha_1 \vdash \alpha} \quad \dots \quad \frac{\Gamma, \alpha_0, \dots, \alpha_{n-1}, \alpha_n \vdash \alpha}{\Gamma, \alpha_0, \dots, \alpha_{n-1} \vdash \alpha} \quad \dots$$

There is a shortest proof of linear size.

Annotating rules and proofs

Now let us annotate the inference rules in the previous example.

When α_i are all given the negative polarity, we have:

$$\frac{\Gamma \vdash \alpha_0}{\Gamma \vdash \alpha_1} \quad \frac{\Gamma \vdash \alpha_0 \qquad \Gamma \vdash \alpha_1}{\Gamma \vdash \alpha_2} \quad \cdots$$

$$\frac{\Gamma \vdash \alpha_0 \qquad \cdots \qquad \Gamma \vdash \alpha_{n-1}}{\Gamma \vdash \alpha_n}$$

The unique proof of $\alpha_0 \vdash \alpha_4$ is annotated by the term:

$$B_4\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0))$$
$$(B_3\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0)))$$

Annotating rules and proofs

Now let us annotate the inference rules in the previous example.

When α_i are all given the negative polarity, we have:

$$\frac{\Gamma \vdash \alpha_0}{\Gamma \vdash \alpha_1} \quad \frac{\Gamma \vdash \alpha_0 \qquad \Gamma \vdash \alpha_1}{\Gamma \vdash \alpha_2} \quad \cdots$$

$$\frac{\Gamma \vdash \alpha_0 \qquad \cdots \qquad \Gamma \vdash \alpha_{n-1}}{\Gamma \vdash \alpha_n}$$

The unique proof of $\alpha_0 \vdash \alpha_4$ is annotated by the term:

$$\begin{gathered} B_4\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0)) \\ (B_3\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0))) \end{gathered}$$

Annotating rules and proofs

Now let us annotate the inference rules in the previous example.

When α_i are all given the negative polarity, we have:

$$\frac{\Gamma \vdash \alpha_0}{\Gamma \vdash \alpha_1} \quad \frac{\Gamma \vdash \alpha_0 \qquad \Gamma \vdash \alpha_1}{\Gamma \vdash \alpha_2} \quad \cdots$$

$$\frac{\Gamma \vdash \alpha_0 \qquad \cdots \qquad \Gamma \vdash \alpha_{n-1}}{\Gamma \vdash \alpha_n}$$

The unique proof of $\alpha_0 \vdash \alpha_4$ is annotated by the term:

$$\begin{gathered} B_4\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0)) \\ (B_3\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0))) \end{gathered}$$

Annotating rules and proofs

Now let us annotate the inference rules in the previous example.

When α_i are all given the negative polarity, we have:

$$\frac{\Gamma \vdash t_0 : \alpha_0}{\Gamma \vdash B_1\, t_0 : \alpha_1} \quad \frac{\Gamma \vdash t_0 : \alpha_0 \qquad \Gamma \vdash t_1 : \alpha_1}{\Gamma \vdash B_2\, t_0\, t_1 : \alpha_2} \quad \cdots$$

$$\frac{\Gamma \vdash t_0 : \alpha_0 \qquad \cdots \qquad \Gamma \vdash t_{n-1} : \alpha_{n-1}}{\Gamma \vdash B_n\, t_0 \cdots t_{n-1} : \alpha_n}$$

The unique proof of $\alpha_0 \vdash \alpha_4$ is annotated by the term:

$$B_4\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0))$$
$$(B_3\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0)))$$

Annotating rules and proofs

Now let us annotate the inference rules in the previous example.

When α_i are all given the negative polarity, we have:

$$\frac{\Gamma \vdash t_0 : \alpha_0}{\Gamma \vdash B_1\, t_0 : \alpha_1} \quad \frac{\Gamma \vdash t_0 : \alpha_0 \qquad \Gamma \vdash t_1 : \alpha_1}{\Gamma \vdash B_2\, t_0\, t_1 : \alpha_2} \quad \cdots$$

$$\frac{\Gamma \vdash t_0 : \alpha_0 \qquad \cdots \qquad \Gamma \vdash t_{n-1} : \alpha_{n-1}}{\Gamma \vdash B_n\, t_0 \cdots t_{n-1} : \alpha_n}$$

The unique proof of $\alpha_0 \vdash \alpha_4$ is annotated by the term:

$$\begin{gathered} B_4\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0)) \\ (B_3\ x_0\ (B_1\ x_0)\ (B_2\ x_0\ (B_1\ x_0))) \end{gathered}$$

Annotating rules and proofs

Now let us annotate the inference rules in the previous example.

When α_i are all given the positive polarity, we have:

$$\frac{\Gamma, \alpha_0, \alpha_1 \vdash \alpha}{\Gamma, \alpha_0 \vdash \alpha} \quad \frac{\Gamma, \alpha_0, \alpha_1, \alpha_2 \vdash \alpha}{\Gamma, \alpha_0, \alpha_1 \vdash \alpha} \quad \cdots$$

$$\frac{\Gamma, \alpha_0, \ldots, \alpha_{n-1}, \alpha_n \vdash \alpha}{\Gamma, \alpha_0, \ldots, \alpha_{n-1} \vdash \alpha}$$

The shortest proof of $\alpha_0 \vdash \alpha_4$ is annotated by the term:

$$\begin{array}{ll} (B_1\ x_0 & (\lambda x_1. \\ (B_2\ x_0\ x_1 & (\lambda x_2. \\ (B_3\ x_0\ x_1\ x_2 & (\lambda x_3. \\ (B_4\ x_0\ x_1\ x_2\ x_3 & (\lambda x_4.\ x_4))))))) \end{array}$$

Annotating rules and proofs

Now let us annotate the inference rules in the previous example.

When α_i are all given the positive polarity, we have:

$$\frac{\Gamma, x_0 : \alpha_0, x_1 : \alpha_1 \vdash t : \alpha}{\Gamma, x_0 : \alpha_0 \vdash B_1 x_0 (\lambda x_1 . t) : \alpha} \quad \frac{\Gamma, x_0 : \alpha_0, x_1 : \alpha_1, x_2 : \alpha_2 \vdash t : \alpha}{\Gamma, x_0 : \alpha_0, x_1 : \alpha_1 \vdash B_2 x_0 x_1 (\lambda x_2 . t) : \alpha} \quad \cdots$$

$$\frac{\Gamma, x_0 : \alpha_0, \ldots, x_{n-1} : \alpha_{n-1}, x_n : \alpha_n \vdash t : \alpha}{\Gamma, x_0 : \alpha_0, \ldots, x_{n-1} : \alpha_{n-1} \vdash B_n x_0 \cdots x_{n-1} (\lambda x_n . t) : \alpha}$$

The shortest proof of $\alpha_0 \vdash \alpha_4$ is annotated by the term:

$$\begin{array}{ll} (B_1\ x_0 & (\lambda x_1. \\ (B_2\ x_0\ x_1 & (\lambda x_2. \\ (B_3\ x_0\ x_1\ x_2 & (\lambda x_3. \\ (B_4\ x_0\ x_1\ x_2\ x_3 & (\lambda x_4.\ x_4)))))))) \end{array}$$

Annotating rules and proofs

Now let us annotate the inference rules in the previous example.

When α_i are all given the positive polarity, we have:

$$\frac{\Gamma, x_0 : \alpha_0, x_1 : \alpha_1 \vdash t : \alpha}{\Gamma, x_0 : \alpha_0 \vdash B_1 x_0 (\lambda x_1 . t) : \alpha} \quad \frac{\Gamma, x_0 : \alpha_0, x_1 : \alpha_1, x_2 : \alpha_2 \vdash t : \alpha}{\Gamma, x_0 : \alpha_0, x_1 : \alpha_1 \vdash B_2 x_0 x_1 (\lambda x_2 . t) : \alpha} \quad \cdots$$

$$\frac{\Gamma, x_0 : \alpha_0, \ldots, x_{n-1} : \alpha_{n-1}, x_n : \alpha_n \vdash t : \alpha}{\Gamma, x_0 : \alpha_0, \ldots, x_{n-1} : \alpha_{n-1} \vdash B_n x_0 \cdots x_{n-1} (\lambda x_n . t) : \alpha}$$

The shortest proof of $\alpha_0 \vdash \alpha_4$ is annotated by the term:

$$\begin{array}{ll} (B_1\ x_0 & (\lambda x_1 . \\ (B_2\ x_0\ x_1 & (\lambda x_2 . \\ (B_3\ x_0\ x_1\ x_2 & (\lambda x_3 . \\ (B_4\ x_0\ x_1\ x_2\ x_3 & (\lambda x_4 .\ x_4)))))))) \end{array}$$

Annotating rules and proofs

Consider the inference rules in the previous example and annotate them.

$$\frac{\Gamma \vdash a_0}{\Gamma \vdash a_1} \qquad \frac{\Gamma \vdash a_0 \quad \Gamma \vdash a_1}{\Gamma \vdash a_2} \qquad \ldots$$

$$\frac{\Gamma \vdash a_0 \quad \cdots \quad \Gamma \vdash a_{n-1}}{\Gamma \vdash a_n}$$

Consider the proofs of $a_0 \vdash a_4$.

Annotating rules and proofs

Consider the inference rules in the previous example and annotate them.

$$\frac{\Gamma \vdash \color{red}{t_0} : a_0}{\Gamma \vdash \color{red}{E_1\, t_0} : a_1} \qquad \frac{\Gamma \vdash \color{red}{t_0} : a_0 \qquad \Gamma \vdash \color{red}{t_1} : a_1}{\Gamma \vdash \color{red}{E_2\, t_0\, t_1} : a_2} \qquad \cdots$$

$$\frac{\Gamma \vdash \color{red}{t_0} : a_0 \qquad \cdots \qquad \Gamma \vdash \color{red}{t_{n-1}} : a_{n-1}}{\Gamma \vdash \color{red}{E_n\, t_0 \cdots t_{n-1}} : a_n}$$

Consider the proofs of $a_0 \vdash a_4$.

Annotating rules and proofs

Consider the inference rules in the previous example and annotate them.

$$\frac{\Gamma \vdash t_0 : a_0}{\Gamma \vdash E_1\, t_0 : a_1} \qquad \frac{\Gamma \vdash t_0 : a_0 \qquad \Gamma \vdash t_1 : a_1}{\Gamma \vdash E_2\, t_0\, t_1 : a_2} \qquad \cdots$$

$$\frac{\Gamma \vdash t_0 : a_0 \qquad \cdots \qquad \Gamma \vdash t_{n-1} : a_{n-1}}{\Gamma \vdash E_n\, t_0 \cdots t_{n-1} : a_n}$$

Consider the proofs of $d_0 : a_0 \vdash t : a_4$.

Annotating rules and proofs

Consider the inference rules in the previous example and annotate them.

$$\frac{\Gamma \vdash t_0 : a_0}{\Gamma \vdash E_1\, t_0 : a_1} \qquad \frac{\Gamma \vdash t_0 : a_0 \quad \Gamma \vdash t_1 : a_1}{\Gamma \vdash E_2\, t_0\, t_1 : a_2} \qquad \cdots$$

$$\frac{\Gamma \vdash t_0 : a_0 \quad \cdots \quad \Gamma \vdash t_{n-1} : a_{n-1}}{\Gamma \vdash E_n\, t_0 \cdots t_{n-1} : a_n}$$

Consider the proofs of $d_0 : a_0 \vdash t : a_4$.

The term annotating the unique proof is

$$\begin{array}{l} (E_4\ (E_3\ \ (E_2\ (E_1\ d_0)\ (E_1\ d_0)) \\ \qquad\qquad (E_2\ (E_1\ d_0)\ (E_1\ d_0))) \\ \qquad (E_3\ \ (E_2\ (E_1\ d_0)\ (E_1\ d_0)) \\ \qquad\qquad (E_2\ (E_1\ d_0)\ (E_1\ d_0)))) \end{array}$$

Annotating rules and proofs

Consider the inference rules in the previous example and annotate them.

$$\frac{\Gamma, a_0, a_1 \vdash A}{\Gamma, a_0 \vdash A} \qquad \frac{\Gamma, a_0, a_1, a_2 \vdash A}{\Gamma, a_0, a_1 \vdash A} \qquad \cdots$$

$$\frac{\Gamma, a_0, \cdots, a_{n-1}, a_n \vdash A}{\Gamma, a_0, \cdots, a_{n-1} \vdash A}$$

Consider the proofs of $a_0 \vdash a_4$.

Annotating rules and proofs

Consider the inference rules in the previous example and annotate them.

$$\frac{\Gamma, x_0 : a_0, x_1 : a_1 \vdash t : A}{\Gamma, x_0 : a_0 \vdash F_1 x_0 (\lambda x_1 . t) : A} \qquad \frac{\Gamma, x_0 : a_0, x_1 : a_1, x_2 : a_2 \vdash t : A}{\Gamma, x_0 : a_0, x_1 : a_1 \vdash F_2 x_0 x_1 (\lambda x_2 . t) : A} \qquad \cdots$$

$$\frac{\Gamma, x_0 : a_0, \cdots, x_{n-1} : a_{n-1}, x_n : a_n \vdash t : A}{\Gamma, x_0 : a_0, \cdots, x_{n-1} : a_{n-1} \vdash F_n x_0 \cdots x_{n-1} (\lambda x_n . t) : A}$$

Consider the proofs of $d_0 : a_0 \vdash t : a_4$.

Annotating rules and proofs

Consider the inference rules in the previous example and annotate them.

$$\frac{\Gamma, x_0 : a_0, x_1 : a_1 \vdash t : A}{\Gamma, x_0 : a_0 \vdash F_1 x_0 (\lambda x_1 . t) : A} \qquad \frac{\Gamma, x_0 : a_0, x_1 : a_1, x_2 : a_2 \vdash t : A}{\Gamma, x_0 : a_0, x_1 : a_1 \vdash F_2 x_0 x_1 (\lambda x_2 . t) : A} \qquad \cdots$$

$$\frac{\Gamma, x_0 : a_0, \cdots, x_{n-1} : a_{n-1}, x_n : a_n \vdash t : A}{\Gamma, x_0 : a_0, \cdots, x_{n-1} : a_{n-1} \vdash F_n x_0 \cdots x_{n-1} (\lambda x_n . t) : A}$$

Consider the proofs of $d_0 : a_0 \vdash t : a_4$.

The term annotating the shortest proof is

$$\begin{array}{ll}
(F_1\ d_0 & (\lambda x_1. \\
(F_2\ d_0\ x_1 & (\lambda x_2. \\
(F_3\ d_0\ x_1\ x_2 & (\lambda x_3. \\
(F_4\ d_0\ x_1\ x_2\ x_3 & (\lambda x_4.\ x_4))))))))
\end{array}$$

Two encodings of untyped λ-terms

Let T be the set $\{\alpha \supset \alpha \supset \alpha, (\alpha \supset \alpha) \supset \alpha\}$ where α is atomic.
We consider $LJ\langle T\rangle$ and only sequents of the form $\alpha, \ldots, \alpha \vdash \alpha$.

Logically, it does not seem interesting.
Once again, we do not care about provability but the structure of proofs.

If α is negative, then we have:

$$\alpha \in \Gamma\ \frac{}{\Gamma \vdash \alpha} \qquad \frac{\Gamma \vdash \alpha \qquad \Gamma \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

If α is positive, then we have:

$$\alpha \in \Gamma\ \frac{}{\Gamma \vdash \alpha} \qquad \{\alpha, \alpha\} \subseteq \Gamma\ \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha \qquad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Two encodings of untyped λ-terms

Let T be the set $\{\alpha \supset \alpha \supset \alpha, (\alpha \supset \alpha) \supset \alpha\}$ where α is atomic.
We consider $LJ\langle T\rangle$ and only sequents of the form $\alpha, \ldots, \alpha \vdash \alpha$.

Logically, it does not seem interesting.
Once again, we do not care about provability but the structure of proofs.

If α is negative, then we have:

$$\frac{}{\Gamma \vdash \alpha}\ \alpha \in \Gamma \qquad \frac{\Gamma \vdash \alpha \quad \Gamma \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

If α is positive, then we have:

$$\frac{}{\Gamma \vdash \alpha}\ \alpha \in \Gamma \qquad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}\ \{\alpha, \alpha\} \subseteq \Gamma \qquad \frac{\Gamma, \alpha \vdash \alpha \quad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Two encodings of untyped λ-terms

Let T be the set $\{\alpha \supset \alpha \supset \alpha, (\alpha \supset \alpha) \supset \alpha\}$ where α is atomic.
We consider $LJ\langle T\rangle$ and only sequents of the form $\alpha, \ldots, \alpha \vdash \alpha$.

Logically, it does not seem interesting.
Once again, we do not care about provability but the structure of proofs.

If α is negative, then we have:

$$\alpha \in \Gamma \; \frac{}{\Gamma \vdash \alpha} \qquad \frac{\Gamma \vdash \alpha \quad \Gamma \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

If α is positive, then we have:

$$\alpha \in \Gamma \; \frac{}{\Gamma \vdash \alpha} \qquad \{\alpha, \alpha\} \subseteq \Gamma \; \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha \quad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Two encodings of untyped λ-terms

Let T be the set $\{\alpha \supset \alpha \supset \alpha, (\alpha \supset \alpha) \supset \alpha\}$ where α is atomic.
We consider $LJ\langle T\rangle$ and only sequents of the form $\alpha, \ldots, \alpha \vdash \alpha$.

Logically, it does not seem interesting.
Once again, we do not care about provability but the structure of proofs.

If α is negative, then we have:

$$\alpha \in \Gamma \ \frac{}{\Gamma \vdash \alpha} \qquad \frac{\Gamma \vdash \alpha \qquad \Gamma \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

If α is positive, then we have:

$$\alpha \in \Gamma \ \frac{}{\Gamma \vdash \alpha} \qquad \{\alpha, \alpha\} \subseteq \Gamma \ \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha \qquad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Two encodings of untyped λ-terms

Let T be the set $\{\alpha \supset \alpha \supset \alpha, (\alpha \supset \alpha) \supset \alpha\}$ where α is atomic.
We consider $LJ\langle T\rangle$ and only sequents of the form $\alpha, \ldots, \alpha \vdash \alpha$.

Logically, it does not seem interesting.
Once again, we do not care about provability but the structure of proofs.

If α is negative, then we have:

$$\alpha \in \Gamma \; \frac{}{\Gamma \vdash \alpha} \qquad \frac{\Gamma \vdash \alpha \qquad \Gamma \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

If α is positive, then we have:

$$\alpha \in \Gamma \; \frac{}{\Gamma \vdash \alpha} \qquad \{\alpha, \alpha\} \subseteq \Gamma \; \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha \qquad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Two encodings of untyped λ-terms

Let T be the set $\{\alpha \supset \alpha \supset \alpha, (\alpha \supset \alpha) \supset \alpha\}$ where α is atomic.
We consider $LJ\langle T\rangle$ and only sequents of the form $\alpha, \ldots, \alpha \vdash \alpha$.

Logically, it does not seem interesting.
Once again, we do not care about provability but the structure of proofs.

If α is negative, then we have:

$$\alpha \in \Gamma \ \frac{}{\Gamma \vdash \alpha} \qquad \frac{\Gamma \vdash \alpha \qquad \Gamma \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

If α is positive, then we have:

$$\alpha \in \Gamma \ \frac{}{\Gamma \vdash \alpha} \qquad \{\alpha, \alpha\} \subseteq \Gamma \ \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha} \qquad \frac{\Gamma, \alpha \vdash \alpha \qquad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Two encodings of untyped λ-terms

α is negative

$$\alpha \in \Gamma \quad \frac{}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma \vdash \alpha \qquad \Gamma \vdash \alpha}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Negative λ-terms

α is positive

$$\alpha \in \Gamma \quad \frac{}{\Gamma \vdash \alpha}$$

$$\{\alpha, \alpha\} \subseteq \Gamma \quad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma, \alpha \vdash \alpha \qquad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Positive λ-terms

Two encodings of untyped λ-terms

α is negative

$$x : \alpha \in \Gamma \quad \frac{}{\Gamma \vdash x : \alpha}$$

$$\frac{\Gamma \vdash \alpha \qquad \Gamma \vdash \alpha}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Negative λ-terms

α is positive

$$\alpha \in \Gamma \quad \frac{}{\Gamma \vdash \alpha}$$

$$\{\alpha, \alpha\} \subseteq \Gamma \quad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma, \alpha \vdash \alpha \qquad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Positive λ-terms

Two encodings of untyped λ-terms

α is negative

$$x : \alpha \in \Gamma \quad \frac{}{\Gamma \vdash x : \alpha}$$

$$\frac{\Gamma \vdash t : \alpha \qquad \Gamma \vdash u : \alpha}{\Gamma \vdash tu : \alpha}$$

$$\frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Negative λ-terms

α is positive

$$\alpha \in \Gamma \quad \frac{}{\Gamma \vdash \alpha}$$

$$\{\alpha, \alpha\} \subseteq \Gamma \quad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma, \alpha \vdash \alpha \qquad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Positive λ-terms

Two encodings of untyped λ-terms

α is negative

$$x : \alpha \in \Gamma \quad \frac{}{\Gamma \vdash x : \alpha}$$

$$\frac{\Gamma \vdash t : \alpha \qquad \Gamma \vdash u : \alpha}{\Gamma \vdash tu : \alpha}$$

$$\frac{\Gamma, x : \alpha \vdash t : \alpha}{\Gamma \vdash \lambda x.t : \alpha}$$

Negative λ-terms

α is positive

$$\alpha \in \Gamma \quad \frac{}{\Gamma \vdash \alpha}$$

$$\{\alpha, \alpha\} \subseteq \Gamma \quad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma, \alpha \vdash \alpha \qquad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Positive λ-terms

Two encodings of untyped λ-terms

α is negative

$$x : \alpha \in \Gamma \quad \frac{}{\Gamma \vdash x : \alpha}$$

$$\frac{\Gamma \vdash t : \alpha \qquad \Gamma \vdash u : \alpha}{\Gamma \vdash tu : \alpha}$$

$$\frac{\Gamma, x : \alpha \vdash t : \alpha}{\Gamma \vdash \lambda x.t : \alpha}$$

Negative λ-terms

α is positive

$$x : \alpha \in \Gamma \quad \frac{}{\Gamma \vdash x : \alpha}$$

$$\{\alpha, \alpha\} \subseteq \Gamma \quad \frac{\Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma, \alpha \vdash \alpha \qquad \Gamma, \alpha \vdash \alpha}{\Gamma \vdash \alpha}$$

Positive λ-terms

Two encodings of untyped λ-terms

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Negative λ-terms

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$$x : \alpha \in \Gamma \quad \frac{}{\Gamma \vdash x : \alpha}$$

$$\{y : \alpha, z : \alpha\} \subseteq \Gamma \quad \frac{\Gamma, x : \alpha \vdash t : \alpha}{\Gamma \vdash t[x \leftarrow yz] : \alpha}$$

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Positive λ-terms

Two encodings of untyped λ-terms

Negative λ-terms $t ::= x \mid tu \mid \lambda x.t$

$\hookrightarrow$ Usual syntax of untyped λ-terms, tree-structure, top-down

Positive λ-terms $t ::= x \mid t[x \leftarrow yz] \mid t[x \leftarrow \lambda y.u]$

$\hookrightarrow$ Allows sharing via explicit substitutions, DAG-structure, bottom-up

We only consider **cut-free** proofs. So what does **cut-elimination** tell us?

$\hookrightarrow$ In both cases, the cut-elimination of *LJF* provides us a natural notion of substitution.

- In the negative case, we get the usual meta-level substitution of untyped λ-calculus.
- In the positive case, we also get a straightforward notion of substitution.

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Related/ongoing work

Proof search in linear logic:

- Focused inverse method by Chaudhuri, a forward proof search method based on the synthetic aspect of focusing. Thanks to the sub-formula property of sequent calculus, we only need to generate sequents only containing sub-formulas of the conclusion, but we can do better thanks to focusing!

Term representation:

- A call-by-value λ-calculus with explicit substitutions called positive λ-calculus can be defined based on positive λ-terms (Wu, APLAS 2023; Accattoli & Wu, MFPS 2024). We are planning to use this rather low-level calculus as an intermediate step between calculi with explicit substitutions and implementation (abstract machines).

Understanding better the polarities of atoms in linear logic.

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Thank you for your listening!